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Imprecise distribution function associated to a random set

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Abstract

Some different extensions to random sets of the most common parameters of a random variable share a common rationale: a random set represents the imprecise observation of a random variable, hence the generalized parameter contains the available information about the respective parameter of the imprecisely observed variable. Following the same principles, in this paper it is proposed a new definition of the distribution function of a random set. This definition is simpler in its formulation and it can be used in more general cases than previous proposals. The properties of the distribution function defined here are discussed: some issues about continuity, convergence of the images of the distribution function, monotonocity and measurability are studied. It is also stated that not all the information conveyed by the random set about the original probability measure (the probability measure induced by the imprecisely observed random variable) is kept by its distribution function.

Keywords: Random set; Distribution function; Dempster's upper and lower probabilities; Sets of probability measures

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1. Introduction

Along with other interpretations (see, for example, [15,23]), random sets are used to model imprecise perceptions of random quantities. In this context, several authors have deal with the generalization of the definitions of the most common parameters associated to random vectors. Thus, we can find in the literature generalizations of the expected value [1,2,10], the median [5,6], the mode [6], the variance [16–18,22], inequality indices [20] and the covariance between two random variables [24]. Some of these definitions [1,5,6,17,18,24] follow a similar scheme: since a random set represents the imprecise observation of a random variable, the new "generalized parameter" contains the available information about the respective parameter (the reader may consult [18] for further explanation on this approach). According to this viewpoint, we have considered in previous papers [3,7] a set of probability measures to represent the available information about the true probability measure governing the random outcome. We study the some important properties about this set of probabilities in connection with the upper and lower probabilities defined by Dempster [11]. We think these studies should not be complete without making a deep investigation about the (imprecise) information the random set provides about the distribution function. In this paper, we start from the initial definition given by Kruse and Meyer [18] and we propose a modification that is simpler in the formulation and can be used in a more general case. The paper is organized as follows. In Section 2, we give an overview of the basic notions about random sets, needed for understanding the rest of the paper. In Section 3 we introduce the new concept, comparing it with the initial definition given by Kruse and Meyer. Afterwards, we study whether the properties of the distribution function of a probability measure are also fulfilled in this context. First, we show that the "random set distribution function" does not contain all the information that Γ provides about the probability measure of the vaguely observed random vector. Then we examine whether the well known properties of upper continuity and convergence of the images of distribution functions to 0 and 1 for divergent sequences of vectors are also satisfied in this general environment for a suitable metric defined on $\mathscr{P}([0,1])$. We prove that some particular conditions are required to the images of Γ to satisfy this properties. On the other hand, the "monotonicity" property of distribution functions is not fulfilled in this context when set-valued arithmetic is used on $\mathscr{P}([0,1])$. Finally, we prove that the random set distribution function is measurable under certain particular conditions for Γ. Section 4 concludes the paper.

2. Preliminary concepts

We will describe a random experiment by a probability space, $(\Omega, \mathscr{A}, P)$, where Ω is the set of all possible outcomes of the experiment, $\mathscr{A}$ is a σ-algebra of subsets of Ω and the set function P, defined on $\mathscr{A}$, is a probability measure. We will represent by a measurable function, $U_0 : \Omega \to \Omega'$, the observation of some attribute of the elements in the referential set, Ω. When our measurement is not totally precise, we do not know the exact value, $U_0(\omega)$, of the characteristics for the individual ω. Hence, we can define a multi-valued mapping, $\Gamma : \Omega \to \mathscr{P}(\Omega')$, that represents the imprecise perception of the measurable function U_0: all we can observe about the point $U_0(\omega)$ is that it belongs to the set $\Gamma(\omega)$. Following the notation established by Kruse and Meyer [18], and Meyer and Kruse [24] we will call U_0 the *original* random variable. A multi-valued mapping Γ is called a *random set* when it is a measurable function with respect to some σ-algebra defined on a certain subset of $\mathscr{P}(\Omega)$. We can find several measurability conditions on the literature (see, for instance [14]). In this paper, we are interested on the probability information provided by the multi-valued mapping, so we need to consider the so-called *strong measurability* condition [14].

Definition 2.1. Let us consider two measurable spaces $(\Omega, \mathscr{A})$ and $(\Omega', \mathscr{A}')$, and a multi-valued map $\Gamma : \Omega \to \mathscr{P}(\Omega')$. Let us also consider, for each $B \in \mathscr{A}'$ the set $\mathscr{C}_B = \{C \subseteq \Omega' | C \cap B \neq \emptyset\}$. We say that Γ is strong measurable when all the sets $\Gamma^{-1}(\mathscr{C}_B) = \{\omega \in \Omega | \Gamma(\omega) \cup B\}, \forall B \in \mathscr{A}'$ are $\mathscr{A}$ measurable. In other words, Γ is strong measurable when it is an $\mathscr{A}$-$\sigma\langle\{\mathscr{C}_B | B \in \mathscr{A}'\}\rangle$ measurable function.

For any measurable subset of Ω', $B \in \Omega'$, the (measurable) sets $\Gamma^{-1}(\mathscr{C}_B)$ and $[\Gamma^{-1}(\mathscr{C}_{B^c})]^c = \{\omega \in \Omega | \Gamma(\omega) \subseteq B\}$ are respectively called the *upper and lower inverse* [25] of B. From now on we will use the simpler notation $B^* = \Gamma^{-1}(\mathscr{C}_B)$ and $B_* = [\Gamma^{-1}(\mathscr{C}_{B^c})]^c$.

In order to summarize the information that the random set Γ contains about the probability measure $P \circ U_0^{-1}$, Dempster [11] defined the *upper* and *lower probabilities* of any measurable set B by the formulae:

$$\begin{aligned} P^*(B) &= P\{\omega \in \Omega | \Gamma(\omega) \cap B \neq \emptyset\} / P\{\omega \in \Omega | \Gamma(\omega) \neq \emptyset\} \\ &= P(B^* | (\Omega')^*) \quad \forall B \in \mathscr{A}', \end{aligned}$$

$$\begin{aligned} P_*(B) &= P\{\omega \in \Omega | \Gamma(\omega) \subseteq B, \Gamma(\omega) \neq \emptyset\} / P\{\omega \in \Omega | \Gamma(\omega) \neq \emptyset\} \\ &= P(B_* | (\Omega')^*) \quad \forall B \in \mathscr{A}'. \end{aligned}$$

Notice that they are well defined for the measurability condition above mentioned. When $[(\Omega')^*]^c = \{\omega \in \Omega | \Gamma(\omega) = \emptyset\}$ is a null set, the equalities $P^*(B) = P(B^*)$, and $P_*(B) = P(B_*)$, hold $\forall B \in \mathscr{A}'$. Through the paper we will assume this condition is satisfied: if Γ is the mathematical model to represent the imprecise observation about the measurable function U_0, $\Gamma(\omega)$ cannot be the empty set, for any $\omega \in \Omega$. At least it must contain the element $U(\omega)$. Under this condition, Dempster's upper and lower probabilities constitute upper and lower bounds for the value $P \circ U_0^{-1}$, i.e. $P_*(B) \leqslant P \circ U_0^{-1}(B) \leqslant P^*(B)$, $\forall B \in \mathscr{A}'$. Hence, some authors (see [23], for instance) describe the information provided by Γ about the "true" probability measure by the pair of dual set functions P_* and P^*. However, it seems to be more precise to consider, for each event $B \in \mathscr{A}'$, the set of possible probability values:

$$P_\Gamma(B) = \{P \circ U^{-1}(B) | U \in S(\Gamma)\},$$

where $S(\Gamma)$ is the class of all *measurable selections* [14] of Γ:

$$S(\Gamma) = \{U : \Omega \to \Omega' | U \text{ measurable}, \ U(\omega) \in \Gamma(\omega) \ \forall \omega \in \Omega\}.$$

It is evident that the set of probability measures "dominated" by P^*:

$$\begin{aligned}\mathscr{M}(P^*) &= \{Q : \mathscr{A}' \to [0,1] \text{ probability} | P_*(B) \leqslant Q(B) \leqslant P^*(B) \ \forall B \in \mathscr{A}'\} \\ &= \{Q : \mathscr{A}' \to [0,1] \text{ probability} | Q(B) \leqslant P^*(B) \ \forall B \in \mathscr{A}'\}\end{aligned}$$

contains the class

$$\varDelta(\Gamma) = \{Q : \mathscr{A}' \to [0,1] \text{ probability} | Q(B) \in P_\Gamma(B) \ \forall B \in \mathscr{A}'\}.$$

On the other hand, it seems to be more operative to work with $\mathscr{M}(P^*)$ and so, it would be interesting to know whether both classes do coincide: on the one hand, we need to know whether $\varDelta(\Gamma)$ is convex. The fact is not true in the general case, as we check in [3,7]. Some authors (see [8,9,19]) observe that the replacement of a set of probabilities by its convex hull may lead to the loss of relevant information. On the other hand, we need to check whether the extremes of $\varDelta(\Gamma)$ coincide with those of $\mathscr{M}(P^*)$. Some studies on that score can be read in [3,4,7,13].

Now we have recalled these preliminary concepts, we can show the definitions and results obtained in this work. In the following section, we will investigate the way to represent the information that Γ provides about the distribution function of the original random variable, F_{U_0}. Thus, we will introduce the concept of "distribution function of a random set", comparing it with the definition introduced by Kruse and Meyer [18]. Then we will examine the relationships between this new information and the information provided by the sets of probabilities $\varDelta(\Gamma)$ and $\mathscr{M}(P^*)$ above described. Afterwards, we will show some interesting properties of the distribution function of a random set.

3. The distribution function for imprecise data

Let us consider a probability space $(\Omega, \mathscr{A}, P)$ and a random vector $U_0 = (U_0^1, \ldots, U_0^k) : \Omega \to \mathbb{R}^k$ that represents the simultaneous observation of k quantitative attributes of the elements in the initial space, Ω. Let the multi-valued mapping $\Gamma : \Omega \to \mathscr{P}(\mathbb{R}^k)$ model the imprecise observation of U_0. The information we have about the distribution function of U_0, F_{U_0}, is given by the set of functions:

$$\mathscr{F}(\Gamma) = \{F_U : \mathbb{R}^k \to [0,1] | U \in S(\Gamma)\}.$$

Since every F_U determines the probability measure $P \circ U^{-1}$, $\mathscr{F}(\Gamma)$ provides all the available information about the "original" probability measure, $P \circ U_0^{-1}$. On the other hand, when we consider a particular vector $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, the information that Γ contains about the value $F_{U_0}(x) = P \circ U_0^{-1}(-\infty, x]$ is determined by the following set of real values:

$$F_\Gamma(x) = \{F_U(x) | U \in S(\Gamma)\} = P_\Gamma((-\infty, x_1] \times \cdots \times (-\infty, x_k]).$$

The last is the most precise set that contains, with complete certainty, the unknown value $F_{U_0}(x)$. The multi-valued mapping, $F_\Gamma : \mathbb{R}^k \to \mathscr{P}([0,1])$ that assigns to each $x \in \mathbb{R}^k$ the value $F_\Gamma(x)$ will be called "the distribution function of Γ".

This new concept is similar to the definition given by Kruse and Meyer [18], but there exist some remarkable differences that we want to point out. Firstly, the authors suppose that the random vector that represents the k-tuple of characteristics of the members of the population studied, U_0, is defined on a product space $(\Omega_1 \times \Omega_2, \mathscr{A}_1 \otimes \mathscr{A}_2, P_1 \otimes P_2)$ where second probability space, $(\Omega_2, \mathscr{A}_2, P_2)$, must satisfy the following condition: for each $\lambda \in [0,1]$, there exists some event $A_2 \in \mathscr{A}_2$ that satisfies the equality $P_2(A_2) = \lambda$. We think that this condition is too much restrictive, since not every random experiment may be described this way: let us think, for instance, on a random choice described by a probability space $(\Omega, \mathscr{A}, P)$, where Ω is a finite population. In this case, $\mathscr{A}$ is a finite class of events, and, therefore, P can only take a finite number of different values. Under the condition imposed by Kruse and Meyer above mentioned, the images of F_Γ are convex as they prove in [18]. This seems a good property, since real intervals are determined by their extreme points. However, in connection wit some studies about non-convex sets of probabilities (see, for instance [8,9,19]), we can observe that the replacement of the set $F_\Gamma(x)$ by its convex hull may lead us to lose important information.

On the other hand, Kruse and Meyer only consider the possibility that the random set may be expressed as the Cartesian product of k real random sets (k characteristics separately observed). This model is not able to represent some usual situations, as we describe below.

Example 3.1. Let us think, for instance, about a population of rectangles where we can only measure their area. The information we obtain about the pair $(B(\omega), H(\omega))$ (base and height) of a particular rectangle ω from the measurement $A(\omega)$ (area of ω) is

$$\Gamma(\omega) = \{(b,h) \in (\mathbb{R}^+)^2 | bh = A(\omega)\} \subseteq \mathbb{R}^2.$$

In general, the information we obtain about a k-tuple of characteristics $(U_0^1, \dots, U_0^k)$ of the individuals of a population from the measurement of a non-injective function of them, $S = g(U_0^1, \dots, U_0^k)$ is given by the set-valued mapping:

$$\Gamma(\omega) = \{(x_1, \dots, x_k) \in \Omega_1 \times \cdots \times \Omega_k | g(x_1, \dots, x_k) \in S(\omega)\} \subseteq \Omega_1 \times \cdots \times \Omega_k,$$

where Ω_i is the set of possible values for the characteristic U_0^i, $i = 1, \dots, k$.

Furthermore, according to Kruse ad Meyer [18], the images of the k components, $\Gamma_1, \dots, \Gamma_k$, of the random set, Γ, need to satisfy the following conditions:

(i) If $\inf \Gamma_i(\omega) > -\infty$, then $\inf \Gamma_i(\omega) \in \Gamma_i(\omega)$.
(ii) If $\sup \Gamma_i(\omega) < \infty$, then $\sup \Gamma_i(\omega) \in \Gamma_i(\omega)$.
(iii) If $\Gamma_i(\omega)$ is not a bounded set, then it must be convex.

In our opinion, these last conditions are very particular.

We can summarize by saying that our definition coincides with Kruse and Meyer's one under the particular hypothesis they impose. But, since we pretend to describe the imprecise observation of a k-tuple of characteristics of elements chosen at random from an arbitrary population, we think that no additional conditions might be imposed to the initial space $(\Omega, \mathscr{A}, P)$ and the images of Γ.

Now we will show some interesting properties of the mapping F_Γ. At first sight, we could think that it is useful to replace the mapping P_Γ mentioned in Section 2 by the distribution function F_Γ, since the last is defined on $\mathbb{R}^k$, while the former is defined on a class of sets. However, contrary to the "classical" case (precise observation described by a random vector) the distribution function does not determine the set function P_Γ, as we show in the following examples.

Example 3.2. Let $\Gamma : \Omega \to \mathscr{P}(\mathbb{R})$ be a random set with closed interval images. In such a case, the real valued mappings $f^*, f_* : \Omega \to \mathbb{R}$ defined as $f_*(\omega) = \min \Gamma(\omega)$, $f^*(\omega) = \max \Gamma(\omega)$, $\forall \omega \in \Omega$ are Borel measurable, (see [13]) and their images are selected from the images of Γ. Hence the images of F_{f_*} and F_{f^*} are contained in those of F_Γ. On the basis of Lemma 3.4 later on enunciated, we can observe that the images of the distribution function of Γ are given by

$$F_\Gamma(x) = \{P(C) | C \in \mathscr{A}, (f^*)^{-1}(-\infty, x] \subseteq C \subseteq (f_*)^{-1}(-\infty, x]\} \quad \forall x \in \mathbb{R}.$$

We can also see that, if the images of a multi-valued mapping $\Gamma' : \Omega \to \mathscr{P}(\mathbb{R})$ have maximum and minimum values and they respectively coincide with the images of f^* and f_*, then it has associated the same distribution function as Γ $(F_\Gamma \equiv F_{\Gamma'})$.

Let us consider, for instance, the measurable space ($\Omega = \{\omega_1, \omega_2\}$, $\mathscr{A} = \mathscr{P}(\Omega)$) provided with the uniform (discrete) probability measure, P, defined on $\mathscr{P}(\Omega)$. Let us also consider the random sets Γ and Γ' defined on Ω by

$$\Gamma(\omega_1) = \{0,1\}, \quad \Gamma(\omega_2) = \{2\}, \quad \Gamma'(\omega_1) = [0,1], \quad \Gamma'(\omega_2) = \{2\}.$$

It is easy to check that they induce the same distribution function ($F_\Gamma = F_{\Gamma'}$). However, Γ contains more precise information than Γ' about the original probability measure. We can observe that there are only two probability measures that are compatible with Γ (since Γ has only two measurable selections), but there exists an infinity of probability measures that are compatible with Γ'.

Example 3.3. Let us now consider two random sets, Γ and Γ', defined on $[0,1]$ as

$$\Gamma(\omega) = \begin{cases} [1,3] & \text{if } \omega \leqslant 1/2, \\ [2,4] & \text{if } \omega > 1/2, \end{cases}$$

$$\Gamma'(\omega) = \begin{cases} [1,4] & \text{if } \omega \leqslant 1/2, \\ [2,3] & \text{if } \omega > 1/2. \end{cases}$$

They have the same distribution function. It is the multi-valued mapping: $F_\Gamma : \mathbb{R} \to \mathscr{P}([0,1])$, $F_\Gamma(x) = [F_{X_2}(x), F_{X_1}(x)]$, $\forall x \in \mathbb{R}$, where X_1 is a random variable that takes the values 1 and 2 with the same probability (1/2) and X_2 takes the values 3 and 4 also with the same probability.

We can observe that, in general, the distribution function of a closed random interval only depends on the marginal distributions of its extremes. However, if we want to know all the information that the random set contains about $P \circ U_0^{-1}$, we need to know the joint distribution of the extremes, $P \circ (X_1, X_2)^{-1}$. In [12] the author proves that, for a closed random interval Γ, the knowledge of $P_*([a,b])$ for all $(a,b) \in \mathbb{R}^2$ determines $P_*(A)$ and $P^*(A)$ for any Borel set $A \in \beta_{\mathbb{R}}$. The same does not happen when we restrict to the class of intervals of the type $(-\infty, x]$, as we have shown in this example.

We deduce from the above examples that the class of probability measures determined by the multi-valued map $F_\Gamma : \mathbb{R}^k \to \mathscr{P}([0,1])$ does not coincide in general with the more precise set $\Delta(\Gamma)$ described in Section 2. We can also compare the information provided by F_Γ about the "true" probability measure $P \circ U_0^{-1}$ with the set of probabilities dominated by P^*, $\mathscr{M}(P^*)$, also described in Section 2. We can deduce from the following example that none of these two models provides, in the general case, more precise information about $P \circ U_0^{-1}$.

Example 3.4. Consider the initial probability space $(\Omega, \mathscr{A}, P)$, where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathscr{A} = \mathscr{P}(\Omega)$ and $P(\{\omega_1\}) = P(\{\omega_2\}) = 0.1$, $P(\{\omega_3\}) = 0.8$. Let us now consider the multi-valued map $\Gamma : \Omega \to \mathscr{P}(\{0,1\})$ given by

$\Gamma(\omega_1)=\{0\}$, $\Gamma(\omega_2)=\{1\}$, $\Gamma(\omega_3)=\{0,1\}$. It is easy to see that the (multi-valued) distribution function of Γ is given by

$$F_\Gamma(x)=\begin{cases}\{0\} & \text{if } x<0,\\ \{0.1,0.9\} & \text{if } x\in[0,1),\\ \{1\} & \text{if } x\geqslant 1.\end{cases}$$

Every probability measure $Q:\mathscr{P}(\{0,1\})\to[0,1]$ compatible with this information satisfies some of the three following conditions:

- $Q(\{0\})=0.1$, $Q(\{1\})=0.9$.
- $Q(\{0\})=0.9$, $Q(\{1\})=0.1$.
- Q has three atoms, $\{0\}$, $\{1\}$ and $\{x\}$, where $x\in(0,1)$, and their probability values are $Q(\{0\})=0.1=Q(\{1\})$ and $Q(\{x\})=0.8$.

On the other hand, we see that the probability measures dominated by the upper probability of Γ, P^*, are those Q satisfying the conditions: $Q(\{0\})=p$, $Q(\{1\})=1-p$, $0.1\leqslant p\leqslant 0.9$.

We observe that none of these two classes of probability measures contains the other.

The distribution function of a random set defined on $\mathbb{R}^k$ preserves, in some sense, some of the properties that characterize the distribution function of a random vector, as we will show below. Some of these properties require the existence of a metric. Since the distribution function of a set-valued function is also set-valued, we need to consider a generalization of the usual metric on $\mathbb{R}$ to the class of subsets of $\mathbb{R}$. We will consider Hausdorff pseudo-metric on $\mathscr{P}(\mathbb{R})$, which is defined as

$$d_{\mathrm{H}}(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|,\sup_{b\in B}\inf_{a\in A}|a-b|\right\}\quad\forall A,B\subseteq\mathbb{R}.$$

If we consider the class of non-empty and compact sets of $\mathbb{R}$, $\mathscr{K}(\mathbb{R}^n)\backslash\{\emptyset\}$, the pair $(\mathscr{K}(\mathbb{R}^n)\backslash\{\emptyset\},d_{\mathrm{H}})$ is a complete and separable metric space.

Before showing the mentioned properties of F_Γ, we need to give some supporting results.

Proposition 3.1 [7]. *Consider the probability space $(\Omega,\mathscr{A},P)$ and a Polish space (E,τ). Let $\Gamma:\Omega\to\mathscr{P}(E)$ be $\mathscr{A}$-$\sigma\langle\{\mathscr{C}_B|B\in\beta_E\}\rangle$ measurable with closed and non-empty images. Then*

$$P^*(A)=\max P_\Gamma(A)\quad\textit{and}\quad P_*(A)=\min P_\Gamma(A)\qquad\forall A\in\mathscr{F}(E)\cup\mathscr{G}(E)$$

($\mathscr{F}(E)$—resp. $\mathscr{G}(E)$—denotes the class of closed—resp. open—sets in (E,τ); β_E denotes the Borel σ-algebra on (E,τ)).

Proposition 3.2 [7]. *Let us consider a probability space* $(\Omega, \mathscr{A}, P)$, *and a normed space* $(E, \|\cdot\|)$. *Let us also consider an* $\mathscr{A}$-$\sigma\langle\{\mathscr{C}_B | B \in \beta_E\}\rangle$ *measurable multi-valued mapping*, $\Gamma : \Omega \to \mathscr{P}(E)$. *Suppose that* $\Gamma(\omega)$ *is open and bounded*, $\forall \omega \in \Omega$ *Then*

$$P^*(A) = \max P_\Gamma(A) \quad \textit{and} \quad P_*(A) = \min P_\Gamma(A) \qquad \forall A \in \beta_E.$$

Lemma 3.3 [7]. *Let us consider a probability space* $(\Omega, \mathscr{A}, P)$, *and a measurable space*, $(\Omega', \mathscr{A}')$. *Let* $\Gamma : \Omega \to \mathscr{P}(\Omega')$ *be a simple random set* (*a random set with a finite number of different images*). *Then*

$$P_*(A) = \max P_\Gamma(A) \quad \textit{and} \quad P^*(A) = P_\Gamma(A) \qquad \forall A \in \mathscr{A}'.$$

Lemma 3.4. *Let us take a probability space* $(\Omega, \mathscr{A}, P)$ *and a measurable space* $(\Omega', \mathscr{A}')$. *Let* $\Gamma : \Omega \to \mathscr{P}(\Omega')$ *be a random set and let us consider an event* $A \in \mathscr{A}'$ *such that* $\{P_*(A), P^*(A)\} \subseteq P_\Gamma(A)$. *Then*

$$P_\Gamma(A) = \{P(C) | C \in \mathscr{A}, A_* \subseteq C \subseteq A^*\}.$$

Proof. By hypothesis, there exist two random selections of Γ, $X_1, X_2 : \Omega \to \mathscr{P}(\Omega')$ such that $P_{X_1}(A) = P_*(A)$ and $P_{X_2}(A) = P(A^*)$. Hence: $A_* \subseteq X_1^{-1}(A)$ with $P(X_1^{-1}(A) \setminus A_*) = 0$, and $X_2^{-1}(A) \subseteq A^*$ where $P(X_2^{-1}(A) \setminus C_2) = 0$. Let us now take an arbitrary $C \in \mathscr{A}$ such that $A_* \subseteq C \subseteq A^*$ and consider the random variable $X_C := X_1 I_{C^c} + X_2 I_C$. It is a measurable selection of Γ. On the other hand, it satisfies the equality $P \circ X_C^{-1}(A) = P(C)$. Hence $P(C)$ belongs to the set of values $P_\Gamma(A)$. $\square$

Lemma 3.5 [25]. *Consider a probability* $(\Omega, \mathscr{A}, P)$, *a measurable space* $(\Omega', \mathscr{A}')$ *and random set* $\Gamma : \Omega \to \mathscr{P}(\Omega')$, *with lower probability* P_*. *Then, for a decreasing sequence of sets* $\{A_n\}_{n\in\mathbb{N}}$, *we have*

$$P_*(A_n) \downarrow P_*\left(\lim_{n\to\infty} A_n\right).$$

The following classical result can be found in [27].

Lemma 3.6. *In a Hausdorff topological space*, (E, τ), *the following statements are satisfied*:

1. *The intersection of a decreasing sequence of compact and non-empty subsets of* E, $\{K_n\}_{n\in\mathbb{N}} \subseteq \mathscr{K}(E) \setminus \{\emptyset\}$, *is also non-empty*.
2. *If* F *is closed and* K *is compact*, *then* $K \cap F$ *is compact*.

Lemma 3.7. *Let us consider a probability space* $(\Omega, \mathscr{A}, P)$, *a Hausdorff topological space* (E, τ) *and its Borel* σ-*algebra*, β_E. *Let* $\Gamma : \Omega \to \mathscr{P}(E)$ *be a random* *set with compact and non-empty images. Let us also consider a decreasing sequence of closed subsets of E, $A_n \downarrow A$. Then $P^*(A_n) \downarrow P^*(A)$.*

Proof. It is sufficient to prove that the sequence $\{A_n^*\}_{n\in\mathbb{N}}$ converges to A^*. On the one hand, it is evident that $A_{n+1}^* \subseteq A_n^*\ \forall n \in \mathbb{N}$. On the other hand, the equality $\bigcap_{n=1}^{\infty} A_n^* = A^*$ holds by Lemma 3.6. $\square$

We also need to prove the following lemma, to get one of the main results in this section. On the other hand, it is important by itself, since it improves a result given in [21].

Lemma 3.8. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and let us consider a normed space $(E, \|\cdot\|)$. Let $\Gamma : \Omega \to \mathscr{P}(E)$ be a random set (measurable with respect to the Borel σ-algebra) with bounded images.*

1. *If $\Gamma(\omega)$ is open, for all $\omega \in \Omega$, then there exists a sequence of simple random sets with compact images, $\{\Gamma_n\}_{n\in\mathbb{N}}$, such that $\Gamma_n(\omega) \subseteq \Gamma_{n+1}(\omega)$, $\forall \omega \in \Omega$, $\forall n \in \mathbb{N}$ and $\Gamma(\omega) = \bigcup_{n\in\mathbb{N}} \Gamma_n(\omega)$, $\forall \omega \in \Omega$.*
2. *If $\Gamma(\omega)$ is closed, for all $\omega \in \Omega$, then there exists a sequence of simple random sets with compact images, $\{\Gamma_n\}_{n\in\mathbb{N}}$, such that, for each $\omega \in \Omega$, there exists $n(\omega) \in \mathbb{N}$ such that $\Gamma_{n+1}(\omega) \subseteq \Gamma_n(\omega)$, $\forall \omega \in \Omega$, $\forall n \geqslant n(\omega)$ and $\Gamma(\omega) = \bigcap_{n \geqslant n(\omega)} \Gamma_n(\omega)$, $\forall \omega \in \Omega$.*

Proof. Let us consider a sequence of positive numbers converging to 0. Since $\overline{B}(0;n)$ is compact, there exists, for each $n \in \mathbb{N}$, a class of sets $\{\overline{B}(x_1^n;\epsilon_n), \ldots, \overline{B}(x_{m_n}^n;\epsilon_n)\}$ such that $\overline{B}(0;n) \subseteq \bigcup_{i=1}^{m_n} \overline{B}(x_i^n;\epsilon_n)$. Let us now consider the class of sets $\{E_1^n, \ldots, E_{m_n}^n\}$, where $E_i^n := \overline{B}(0;n) \cap \overline{B}(x_i^n;\epsilon_n)$.

Let us define, for each $n \in \mathbb{N}$, the class $\mathscr{C}_n = \{A_1^n, \ldots, A_{k_n}^n\}$ as follows:

$$\begin{aligned}
&k_1 = m_1,\\
&A_i^1 := E_i^1 \quad \forall i = 1, \ldots, m_1,\\
&B_i^n := E_i^n \cap (B(0;n-1))^c \quad \forall i = 1, \ldots, m_n,\ \ n \geqslant 2,\\
&C_{ij}^n := E_i^n \cap A_j^{n-1} \quad \forall i = 1, \ldots, m_n,\ \ j = 1, \ldots, m_{n-1},\ \ n \geqslant 2,\\
&\mathscr{C}_n = \{A_1^n, \ldots, A_{k_n}^n\},\\
&\quad := \{B_1^n, \ldots, B_{m_n}^n, C_{11}^n, \ldots, C_{1m_{n-1}}, \ldots, C_{m_n 1}^n, \ldots, C_{m_n m_{n-1}}^n\} \quad \forall n \geqslant 2.
\end{aligned}$$

This family of sets satisfies the conditions $\delta(A_i^n) \leqslant \epsilon_n$, $i = 1, \ldots, k_n$, $n \in \mathbb{N}$ and $\bigcup_{i=1}^{k_n} A_i^n = \overline{B}(0;n)$.

1. Consider, for each $n \in \mathbb{N}$, the measurable set-valued function $\Gamma_n : \Omega \to \mathscr{P}(E)$ defined as $\Gamma_n(\omega) = \bigcup_{\{i\in\{1,\ldots,m_n\} | A_i^n \subseteq \Gamma(\omega)\}} A_i^n$. We observe that it has a finite number of different images (it is a simple multi-valued mapping).

Let us now prove that, for each $\omega \in \Omega$, the sequence of sets $\{\Gamma_n(\omega)\}_{n\in\mathbb{N}}$ is increasing: if $x \in \Gamma_n(\omega)$, then there exists some index $i_n \in \{1,\dots,k_n\}$, such that $x \in A^n_{i_n} \subseteq \overline{B}(0;n)$. Since $\overline{B}(0;n+1) = \bigcup_{i=1}^{m_{n+1}} E_i^{n+1}$, we have $A^n_{i_n} = \bigcup_{i=1}^{m_{n+1}} (A^n_{i_n} \cap E_i^{n+1}) = \bigcup_{i=1}^{m_{n+1}} C^{n+1}_{i_n i}$. Therefore, there exists some index $j_n \in \{1,\dots,m_n\}$ such that $x \in C^{n+1}_{i_n j_n} \in \mathscr{C}_{n+1}$. Furthermore, $C^{n+1}_{i_n j_n} \subseteq A^n_{i_n} \subseteq \Gamma(\omega)$. Thus, $x \in \Gamma_{n+1}(\omega)$. On the other hand, according to the definition of $\{\Gamma_n\}_{n\in\mathbb{N}}$, it is evident that $\Gamma_n(\omega) \subseteq \Gamma(\omega)$, $\forall \omega \in \Omega, n \in \mathbb{N}$. Finally, let us prove that, for each $\omega \in \Omega$ we have that $\Gamma(\omega) \subseteq \bigcup_{n\in\mathbb{N}} \Gamma_n(\omega)$. For arbitrary $\omega \in \Omega$ and $x \in \Gamma(\omega)$, since $\Gamma(\omega)$ is open and bounded, there exist $n_1 \in \mathbb{N}$ such that $\Gamma(\omega) \subseteq \overline{B}(0;n_1)$ and $\epsilon > 0$ such that $B(x;\epsilon) \subseteq \Gamma(\omega)$. Let us take some $n_2 \in \mathbb{N}$ so that $\epsilon_n < \epsilon/2$, $\forall n \geqslant n_2$. Let $n_0 := \max\{n_1, n_2\}$. Under these conditions, we know that there exists $i_{n_0} \in \{1,\dots,k_{n_0}\}$ such that $x \in A_{i_{n_0}} \subseteq \Gamma(\omega)$. Thus, $x \in \Gamma_{n_0}(\omega)$.

2. Let us consider, for each $n \in \mathbb{N}$, the set-valued function $\Gamma_n : \Omega \to \mathscr{P}(E)$ defined as $\Gamma_n(\omega) = \bigcup_{\{i\in\{1,\dots,m_n\} \mid A_i^n \cap \Gamma(\omega) \neq \emptyset\}} A_i^n$. We observe that it is simple and measurable. Following a similar scheme as in part 1, we can see that this sequence satisfies the required conditions. □

On the basis of the previous results, we can investigate whether the well known properties that characterize the distribution functions of random vectors are fulfilled in this general context. First we will study the limits and the right continuity.

Proposition 3.9. *Let us consider the probability space $(\Omega, \mathscr{A}, P)$. Let $\Gamma : \Omega \to \mathscr{P}(\mathbb{R}^k)$ be a random set with closed images. Then, the following properties are fulfilled*:

1. (Limits) *If the images of Γ are bounded subsets of $\mathbb{R}^k$ a.s.(P) then*:
   - $\lim_{x\to-\infty} d_{\mathrm{H}}(F_\Gamma(x_1,\dots,x,\dots,x_k), \{0\}) = 0.$
   - $\lim_{x_1\to\infty,\dots,x_k\to\infty} d_{\mathrm{H}}(F_\Gamma(x_1,\dots,x_k), \{1\}) = 0.$
2. (Right-continuity) *If the images of Γ are compact*, *then*

$$\lim_{x\downarrow a} d_{\mathrm{H}}(F_\Gamma(x), F_\Gamma(a)) = 0.$$

Proof. 1. Consider, for each $i \in \{1,\dots,k\}$ the measurable set-valued function $\Gamma_i = \Pi_i \circ \Gamma$, where Π_i is the ith projection. Let us also consider the measurable real function $f_i^* := \sup \Gamma_i$. Under the hypotheses of this theorem, each function f_i^* is bounded a.s.(P). So, the k-dimensional random vector $f^* = (f_1^*,\dots,f_k^*)$ is also bounded. Hence, $\lim_{x\to-\infty} F_{f^*}(x_1,\dots,x,\dots,x_k) = 0$. Since $F_{f^*}(x) \geqslant \sup F_\Gamma(x)$ $\forall x \in \mathbb{R}^k$, we obtain that

$$\lim_{x\to-\infty} d_{\mathrm{H}}(F_\Gamma(x_1,\dots,x,\dots,x_k), \{0\}) = \lim_{x\to-\infty} \sup F_\Gamma(x_1,\dots,x,\dots,x_k) = 0.$$

The proof is similar to the last one.

2. Let us take a decreasing (for the partial usual order in $\mathbb{R}^k$) sequence $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathbb{R}^k$, that converges to a vector a, $x_m \downarrow a$. For an arbitrary $m \in \mathbb{N}$, take an arbitrary $x \in P_\Gamma((-\infty, x_m])$. We know that $x = P(C)$ with $C \in \mathscr{A}$ and $(-\infty, x_m]_* \subseteq C \subseteq (-\infty, x_m]^*$. Since $(-\infty, a]_* \subseteq C \cap (-\infty, a]^* \subseteq (-\infty, a]^*$, we know by Proposition 3.1 and Lemma 3.4, that $P(C \cap (-\infty, a]^*) \in P_\Gamma((-\infty, a])$. On the other hand, $C \subseteq (-\infty, x_m]^*$ and $(-\infty, a]^* \subseteq (-\infty, x_m]^*$, so we see that $P(C \cap ((-\infty, a]^*)^c) \leqslant P^*((-\infty, x_m]) - P^*((-\infty, a])$. Hence we obtain that $\inf_{y\in P_\Gamma((-\infty,a])} |x - y| \leqslant P^*((-\infty, x_m]) - P^*((-\infty, a])$. Similarly, for an arbitrary $y \in P_\Gamma((-\infty, a])$, we can see that $\inf_{x\in P_\Gamma((-\infty,x_m])} |x - y| \leqslant P_*((-\infty, x_m]) - P_*((-\infty, a])$. Since $\lim_{m\to\infty} P_*((-\infty, x_m]) = P_*((-\infty, a])$ (see Lemma 3.5) and $\lim_{m\to\infty} P^*((-\infty, x_m]) = P^*((-\infty, a])$, we immediately deduce the thesis of this result. □

We can see that F_Γ does not necessarily fulfill the d_{H}-right continuity property when we do not impose the condition that Γ has compact images, as we observe in the following example.

Example 3.5. Consider the probability space $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$, where $\beta_{[0,1]}$ is the restriction to $[0,1]$ of the Borel σ-algebra induced by the usual topology on $\mathbb{R}$, and $\lambda_{[0,1]}$ is the restriction of Lebesgue measure to the same interval. Consider the constant set-valued mapping $\Gamma : [0,1] \to \mathscr{P}(\mathbb{R})$ defined as $\Gamma(x) = (0,1), \forall x \in [0,1]$. Consider the point $a = 0$ and a decreasing sequence of values $\{x_n\}_{n\in\mathbb{N}}$, $x_n > 0$, $\forall n \in \mathbb{N}$ converging to 0. Under these conditions, we observe that $F_\Gamma(x_n) = [0,1]$, $\forall n \in \mathbb{N}$. But, on the other hand $F_\Gamma(0) = \{0\}$, so $\lim_{n\to\infty} d_{\mathrm{H}}(F_\Gamma(x_n), F_\Gamma(0)) = 1 \neq 0$.

On the other hand, we can check that the "extended monotonicity" property of distribution functions is not fulfilled in this more general case. Let us consider the set-valued mapping $\Delta F_\Gamma : \mathrm{Rec}(\mathbb{R}^k) \to \mathscr{P}(\mathbb{R})$ defined as $\Delta F_\Gamma(a,b) := \oplus_{i=1}^k \gamma_i \odot F_\Gamma(v_i)$ (set-valued arithmetic), where the v_is are the 2^n vertices of the rectangle determined by a and b, and $\gamma_i := (-1)^r$ for the vertices with r maximum coordinates and $k - r$ minimum coordinates, $\forall a, b \in \mathbb{R}^k$ such that $a \leqslant b$, where "$\leqslant$" is the usual partial order defined in $\mathbb{R}^k$. Then, the images of ΔF_Γ are not necessarily contained in $\mathbb{R}^+$. In the "classical" case of random vectors (precise observation of the characteristics), the images of F_Γ are positive numbers (see, for instance [26]).

It is well known in probability theory that the distribution function of a random vector is measurable. Next, we will show that the distribution function of a random set is also measurable, under some particular conditions.

Theorem 3.10. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\Gamma : \Omega \to \mathscr{P}(\mathbb{R}^k)$ a simple random set with $\beta_{\mathbb{R}^k}$-measurable images. Then the set-valued mapping F_Γ is also measurable.*

Proof. Let us suppose that Γ can be expressed as $\Gamma = \oplus_{i=1} P_n A_i \odot I_{C_i}$, where $C_i \in \mathscr{A}$ (this condition is satisfied if Γ is $\mathscr{A}$-$\sigma\langle\mathscr{C}(\beta_{\mathbb{R}^k})\rangle$ measurable) and A_i is measurable, $\forall i = 1, \ldots, n$. Now consider the family of sets $\Pi := \{\bigcap_{i=1}^n B_i | B_i \in \{A_i, A_i^c\}\ \forall i = 1, \ldots, n\}$. Let us denote by $D_1, \ldots, D_r$ the elements of Π. This family of sets is a partition of $\mathbb{R}^k$. Let us consider an arbitrary $i_0 \in \{1, \ldots, r\}$ and two arbitrary elements (in case of their existence) of it x, y. If D_{i_0} does not have maximum and minimum or else x and y do not coincide with them, then a simple but large proof leads us to see that $F_\Gamma(x) = F_\Gamma(y)$. Thus, for each $x \in \mathbb{R}^k$ the set $R(x) = \{y \in \mathbb{R}^k | F_\Gamma(y) = F_\Gamma(x)\}$ belongs to the set of (finite) unions and intersections of the elements of the family: $\{D_1, \ldots, D_r\} \cup \bigcup_{i=1}^r \{\{x\} | x \in \{\min D_i\}, \max D_i\}$. Since the elements of this class are $\beta_{\mathbb{R}^k}$-measurable, the former set, $R(x)$ is also measurable. We also see that F_Γ has a finite quantity of different images.

To see that F_Γ is measurable, we need to prove that, for each $S \in \beta_{\mathbb{R}}$, the set $\{x \in \mathbb{R}^k | F_\Gamma(x) \cap S \neq \emptyset\}$ is $\beta_{\mathbb{R}^k}$-measurable. We have proved that F_Γ is simple, so it suffices to prove that for an arbitrary Borel set $D \in \beta_{\mathbb{R}}$, $\{x \in \mathbb{R}^k | F_\Gamma(x) = D\}$ is $\beta_{\mathbb{R}^k}$-measurable. This is true according to the arguments in last paragraph. □

Theorem 3.11. *Let $(\Omega, \mathscr{A}, P)$ be a probability space. Let $\Gamma : \Omega \to \mathscr{P}(\mathbb{R}^k)$ be a random set with open and bounded images. Then, $F_\Gamma : \mathbb{R}^n \to \mathscr{P}([0,1])$ is measurable.*

Proof. By Proposition 3.2 and Lemma 3.4, we have that $P_\Gamma(-\infty, x] = \{P(C) | C \in \mathscr{A} | (-\infty, x]_* \subseteq C \subseteq (-\infty, x]^*\}$. On the other hand, by Lemma 3.8, we know that there exists an increasing (in Kuratowski sense) sequence of simple random sets, $\{\Gamma_n\}_{n\in\mathbb{N}}$, such that $\Gamma(\omega) = \bigcup_{n=1}^\infty \Gamma_n(\omega)$, $\forall \omega \in \Omega$. By Lemma 3.3, we obtain that $P_n^*((-\infty, x]) = \max P_\Gamma((-\infty, x])$, and $P_{n*}((-\infty, x]) = \min P_\Gamma((-\infty, x])$, where P_n^* and P_{n*} are respectively the upper and lower probabilities of Γ_n. Hence, we have that $P_{\Gamma_n}((-\infty, x]) = \{P(C) | A_{n*} \subseteq C \subseteq A_n^*\}$, where A_{n*} denotes the measurable set $\{\omega \in \Omega | \Gamma_n(\omega) \subseteq (-\infty, x]\}$ and A_n^* is $\{\omega \in \Omega | \Gamma_n(\omega) \cap (-\infty, x] \neq \emptyset\}$. Furthermore, we can easily see that $P^*((-\infty, x]) = \lim_{n\to\infty} P(A_n^*)$ and $P_*((-\infty, x]) = \lim_{n\to\infty} P(A_{n*})$. Let us prove now that $\lim_{n\to\infty} d_{\mathrm{H}}(P_\Gamma((-\infty, x]), P_{\Gamma_n}((-\infty, x])) = 0$. Let us take an arbitrary $y_n \in F_{\Gamma_n}(x)$. Then $y_n = P(C)$, with $A_{n*} \subseteq C \subseteq A_n^*$. We can also observe that $A_* \subseteq A_{n*} \subseteq C \subseteq A_n^* \subseteq A^*$, so $P(C) \in P_\Gamma(A)$ and then $\inf_{z \in P_\Gamma(A)} |y_n - z| = 0$. We can also observe that, for an arbitrary $z \in P_\Gamma(A)$, $\inf_{y_n \in P_{\Gamma_n}(A)} |z - y_n| \leqslant \epsilon(n)$, where $\{\epsilon(n)\}_{n\in\mathbb{N}}$ converges to 0.

Thus, F_Γ is the punctual limit (in d_{H} or Kuratowski sense) of the sequence of set-valued mappings $\{F_{\Gamma_n}\}_{n\in\mathbb{N}}$. On the other hand, each set-valued mapping $F_{\Gamma_n} : \mathbb{R}^k \to \mathscr{P}([0,1])$ is simple since Γ_n is. We also know that it is measurable, by Theorem 3.10 (the images of each F_{Γ_n} are $\beta_{\mathbb{R}^k}$-measurable, since they are compact. Hence, we haven proved that F_Γ is measurable. □

4. Concluding remarks

Here, we have checked the measurability of F_Γ in very particular cases. As an open problem, we plan to prove it in a more general case. A similar proof to that of Theorem 3.11 would not apply in a different case: in [7] we have provided some results related to the equality $P^* = \sup \mathscr{P}(\Gamma)$ under different conditions for the images of Γ. However, in the general case we cannot get an "increasing" sequence of simple random sets that converges to Γ. So, we should verify the measurability of F_Γ with different techniques.

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