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Ordering finitely generated sets and finite interval-valued hesitant fuzzy sets

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Abstract

Ordering sets is a long-standing open problem due to its remarkable importance in many areas such as decision making, image processing or human reliability. This work is focused on introducing methods for ordering finitely generated sets as a generalization of those methods previously defined for ordering intervals. In addition, these orders between finitely generated sets are also improved to present orders between finite interval-valued hesitant fuzzy sets. Finally, finite interval-valued hesitant fuzzy preference relations are introduced and used to define a new order between finite interval-valued hesitant fuzzy sets.

Key words:
Linear order, interval, finitely generated set, finite interval-valued hesitant fuzzy set, finite interval-valued hesitant fuzzy preference relation.

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1 Introduction

Since Zadeh introduced fuzzy sets to model the uncertainty associated to the concept of imprecision ([36]), several extensions of fuzzy sets have been introduced: interval-valued fuzzy sets ([13,27,37]), Atanassov's intuitionistic fuzzy sets ([1,2]), hesitant fuzzy sets ([18,25,30]), typical hesitant fuzzy sets ([5]), fuzzy sets of type 2 ([38]), etc. In particular, interval-valued fuzzy sets have been deeply studied. For example Zhang et al. develop in [40] an adjustable approach to interval-valued intuitionistic fuzzy soft sets and define the concept of weighted interval-valued intuitionistic fuzzy soft set. In [24] the problem of the interval-valued fuzzy sets synthesis is studied.

All these extensions have received an increasing interest in different fields such as classification ([14,28]), human reliability ([23]), image processing ([7]). In particular, there are a lot of works focused on solving decision making problems using extensions of fuzzy sets. We highlight the works developed in [32], where the aggregation of hesitant fuzzy information is studied. In addition, in [39] new aggregation operators are utilized to develop techniques for multiple attribute group decision making with hesitant fuzzy information. Liu and Sun in [21] develop a generalized power hesitant fuzzy ordered weighted average operator to aggregate hesitant fuzzy numbers. Finally, Chen et al. in [11] introduce interval-valued hesitant preference relations to describe uncertain evaluation information in group decision making processes.

Therefore, an order between the objects of these extensions is necessary to properly implement these applications. Several orders have been studied and defined between fuzzy sets during the last 20 years ([10,12,20,22,35]).

Regarding ordering for interval-valued fuzzy sets, Barrenechea et al. in [3] develop a construction method for interval-valued fuzzy preference relations from a fuzzy preference relation. They represent the lack of knowledge or ignorance that experts suffer when they define the membership values of the elements of that fuzzy preference relation. In addition, they propose a generalization of Orlovsky's non dominance method to solve decision making problems using interval-valued fuzzy preference relations. Bustince et al. in [9] address the problem of choosing a total order between intervals. Their procedure is based on studying firstly the additivity of interval-valued aggregation functions. Then, they treat the problem of preserving admissible orders by linear transformations. Finally they study the construction and properties of interval-valued ordered weighted aggregation operators by means of admissible orders.

Nevertheless, despite this kind of sets are reaching the spotlight in recent years, orders between hesitant fuzzy sets (and their extensions) have not been deeply

explored yet. Actually, to the best of our knowledge, literature about orders between hesitant fuzzy sets (and their extensions) is sparse, even if there exists some works which have dealt with orders for typical hesitant fuzzy-sets ([4,5]).

The goal of this paper is twofold. Firstly, finite interval-valued fuzzy sets, which are a natural extension of typical hesitant fuzzy sets, are introduced. This kind of sets could immediately derive on lots of applications in several fields such as, for example, group decision making. For instance, this new kind of sets could model, at the same time, experts and criteria.

Furthermore, as those new sets are defined by a membership function which is the union of disjoint closed intervals (finitely generated sets), the study of these finitely generated sets turns into one of the key points of this paper. Therefore, this kind of sets is deeply analysed and the concept of α^{sg}-point is introduced. This α^{sg}-point measures, according to a parameter, the degree of optimism adopted when comparing finitely generated sets.

On the other hand, when using finite interval-valued hesitant fuzzy sets in group decision making, we will need to define a way of comparing these sets. Hence, several orders between finite interval-valued hesitant fuzzy sets are introduced in the last sections of this paper. To that end, α^{sg}-projections and finite interval-valued hesitant fuzzy preference relations are introduced.

The structure of this paper is as follows: Section 2 gives an overview of the preliminary definitions used in this paper. In addition, some methods for ordering real intervals are also reviewed in order to improve them for ordering finitely generated sets in Section 3. Section 4 is devoted to construct several orders between finite interval-valued hesitant fuzzy sets. Finally, some conclusions and open problems are analysed in Section 5.

2   Preliminary definitions

This section is devoted to briefly introduce several well-known basic concepts and to fix the notations used in this paper.

2.1   Fuzzy sets and their extensions

Definition 1 *([36]) A fuzzy set A over X is an object:*

$$A = \{(x, \mu_A(x)) | x \in X\},$$

where $\mu_A : X \to [0,1]$ is called membership function.

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The set of all ordinary fuzzy sets that can be defined on the universe $[0, 1]$ *is denoted by* $F([0, 1])$.

By abuse of notation, in the literature the membership function is frequently denoted by A insted of μ_A.

In some cases, the uncertainty measured by the fuzzy sets is not enough or it does not fit with the nature of the problem. In other cases, it is not possible to find an accurate way to define the membership functions. In these cases, it is common to make use of the so-called extensions of fuzzy sets. The most relevant ones are defined below.

Definition 2 *([1]) An Atanassov's intuitionistic fuzzy set A on the universe X is defined as*

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\},$$

where $\mu_A, \nu_A : X \to [0, 1]$ satisfy

$$\mu_A(x) + \nu_A(x) \leq 1, \ \forall x \in X.$$

Here μ_A and ν_A define, respectively, the degree of membership and the degree of non-membership of the element x to the set A.

Definition 3 *([27]) An interval-valued fuzzy set A on the universe X is defined by a mapping*

$$A : X \to L([0, 1]),$$

such that the membership degree of $x \in X$ is given by $A(x) = [\underline{A}(x), \overline{A}(x)] \in L([0, 1])$, where $\underline{A} : X \to [0, 1]$ and $\overline{A} : X \to [0, 1]$ are, respectively, mappings defining the lower and the upper bound of the membership interval $A(x)$ and $L([0, 1])$ denotes the set of all closed subintervals in $[0, 1]$. The class of all interval-valued fuzzy sets on X is denoted by $IVFS(X)$.

In [1], it is proven that Atanassov's intuitionistic fuzzy sets are equivalent to interval-valued fuzzy sets and we can work with either obtaining the same results. However, conceptually do not model the same problem ([29]). Both are commonly used in literature and the one which suits better with the nature of the problem is usually utilized.

We can further generalize fuzzy sets allowing the membership degrees to be another fuzzy set.

Definition 4 *([38]) A fuzzy set of type 2 A on the universe X is defined by a mapping*

$$A : X \to F([0, 1]).$$

Remark 1 *Note that interval-valued fuzzy sets are a particular case of fuzzy sets of type 2, where the membership degree of each element is given by the*

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characteristic function of a closed subinterval of [0,1].

Another extension may be defined for which the membership of the elements of the set could be any subset of the interval $[0, 1]$.

Definition 5 *([30]) A hesitant fuzzy set A on the universe X is given by*

$$A = \{(x, \mu_A(x)) | x \in X\},$$

where the membership function takes values in the power crisp set of $[0, 1]$. We will denote the set of all hesitant fuzzy sets on X by $H(X)$.

Remark 2 *In fact, hesitant fuzzy sets were already introduced in 1976 by Grattan-Guinness in [18] with the name of set-valued fuzzy sets.*

For most applications, a specific kind of hesitant fuzzy sets is usually utilized. The membership functions of this subfamily of hesitant fuzzy sets are not any subset in $[0, 1]$, we reduce to the case where we have a finite union of singletons.

Definition 6 *([4,5]) A typical hesitant fuzzy set A on the universe X is a hesitant fuzzy set where for each $x \in X$, $\mu_A(x)$ can be expressed as a finite union of singletons in $[0, 1]$. We will denote the set of all typical hesitant fuzzy sets on X by $TH(X)$.*

However, in this paper, we introduce a generalization of this kind of sets considering closed intervals instead of singletons.

Definition 7 *A finite interval-valued hesitant fuzzy set A on the universe X is given by*

$$A = \{(x, \mu_A(x)) | x \in X\},$$

where, for each $x \in X$, the membership function $\mu_A(x)$ can be expressed as a finite union of disjoint closed intervals in $[0, 1]$. We will denote the set of all finite interval-valued hesitant fuzzy sets on X by $FIVH(X)$.

Note that typical hesitant fuzzy sets are a particular case of finite interval-valued hesitant fuzzy sets when the closed intervals are restricted to be singletons. Furthermore, finite interval-valued hesitant fuzzy sets are a particular case of hesitant fuzzy sets when the membership functions are restricted to be finite unions of disjoint closed intervals.

It must be remarked that finite interval-valued hesitant fuzzy sets are not the same defined by Chen et al. in [11]. In that case, as the interval intersection is not empty, the membership functions are not formally defined as a subset of $[0, 1]$. Moreover, as they are different types of sets, they will model different kinds of real life problems. For instance, in a group decision making problem, on the one hand, Chen et al.'s sets can model a problem where several members

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of a family express their preferences on a set of candidates. On the other hand, finite interval-valued hesitant fuzzy sets can model a problem where preferences, for any reason, are not given in a convex form. Such an example can happen when both alternatives are known to be non-indifferent but we do not know which of the two alternatives is preferred to the other.

Furthermore, finiteness cannot be considered as a mere constraint on the whole set because, as we know, there are lot of properties easily applied to finite sets that are not translatable to infinite sets.

These finite unions of disjoint closed intervals are going to be called finitely generated sets and are going to be deeply analysed below.

Definition 8 *The class of n-finitely generated sets in the interval $[0,1]$ is:*

$$FG_n([0,1]) = \{J \subseteq [0,1] | J = I_1 \cup \ldots \cup I_n \text{ for some disjoint } I_1, \ldots, I_n \in L([0,1])\},$$

where $L([0,1])$ is the set of all closed subintervals in $[0,1]$.

Definition 9 *The class of finitely generated sets in the interval $[0,1]$ is:*

$$FG([0,1]) = \bigcup_{n=1}^{+\infty} FG_n([0,1]).$$

Remark 3 $L([0,1]) = FG_1([0,1]) \subseteq FG([0,1])$.

Moreover, it is trivial to prove the following proposition.

Proposition 1 *Let $I \subseteq [0,1]$. Then, the following statements are equivalent:*

- $I \in FG([0,1])$.
- *There exists a unique value $n \geq 1$ such that $I \in FG_n([0,1])$.*

Remark 4 *Let $I \in FG([0,1])$. As a consequence of Proposition 1, the unique value $n \geq 1$ such that $I \in FG_n([0,1])$ is denoted by n_I.*

By definition, it is also trivial to prove the following equivalence.

Proposition 2 *Let $A \in H(X)$. Then, the following statements are equivalent:*

- *A is a finite interval-valued hesitant fuzzy set.*
- $\mu_A(x) \in FG([0,1]) \ \forall x \in X$.

Once finite interval-valued hesitant fuzzy sets and finitely generated sets are introduced, we study how an order of finite interval-valued hesitant fuzzy sets could be defined. For this purpose, we need to start by introducing some methods to order finite generated sets and, for that, we will recall several methods to order real intervals (see [8,34]).

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2.2 Ordering intervals

This section focuses on the most widespread methods to order real intervals. We will recall their definitions and main relationships among them.

2.2.1 Bounds comparison

The comparison of the bounds can be done in four different ways.

Definition 10 *([15]) Let $A = [a_1, a_2]$ y $B = [b_1, b_2]$ two real intervals, then*

*(1) B **strongly dominates** A iff $b_1 > a_2$.*
*(2) B **maxi-min dominates** A iff $b_1 > a_1$.*
*(3) B **maxi-max dominates** A iff $b_2 > a_2$.*
*(4) B **weakly dominates** A iff $b_2 > a_1$.*

Note that condition (1) implies all the others and condition (4) is implied by all the others. However, it is not possible to establish an implication between conditions (2) and (3).
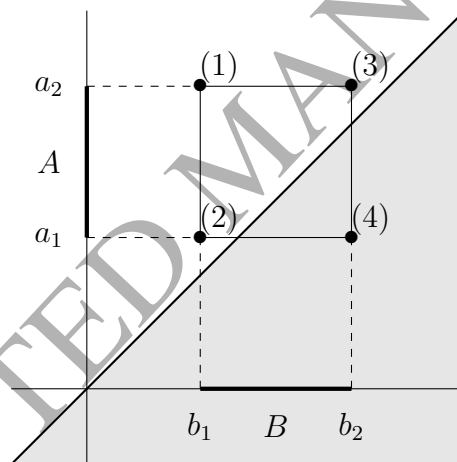

Fig. 1. Graphical interpretation of Bounds comparison.

Fig. 1 shows a graphical interpretation of these orders. It presents the rectangle generated by the Cartesian product of the two intervals A and B. Each vertex of the rectangle represents one of the four ways of dominance defined by this method. According to the graphical interpretation, we can say that a dominance holds if its respective vertex is below the straight line $x = y$. For example, in the particular case of Fig. 1 only condition 4 holds.

In literature, Bustince et al. ([8]) proposed an improved version of maxi-min and maxi-max dominance, which constructs linear orders. These improved versions are called lexicographical orders and are defined as follows:

Definition 11 *([8]) Let $A = [a_1, a_2]$ y $B = [b_1, b_2]$ two real intervals, then*

$$A \leq_{Lex1} B \quad \Leftrightarrow \quad a_1 < b_1 \ or \ (a_1 = b_1 \ and \ a_2 \leq b_2).$$

$$A \leq_{Lex2} B \quad \Leftrightarrow \quad a_2 < b_2 \ or \ (a_2 = b_2 \ and \ a_1 \leq b_1).$$

Besides, observe that strong domination does not define an order, because in general, it is not true that $[a, b] \leq [a, b]$.

Analogously, weak domination does not define an order since in this case, transitivity does not hold $[0.6, 0.7] < [0, 1]$ and $[0, 1] < [0.2, 0.5]$, but $[0.6, 0.7]$ is not smaller than $[0.2, 0.5]$.

2.2.2 Midpoint comparison

This method consist in comparing the midpoint of the two intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$:

Definition 12 *([19]) Let $A = [a_1, a_2]$ y $B = [b_1, b_2]$ two real intervals, then*

$$A \leq_M B \quad \Leftrightarrow \quad a_1 + a_2 \leq b_1 + b_2.$$
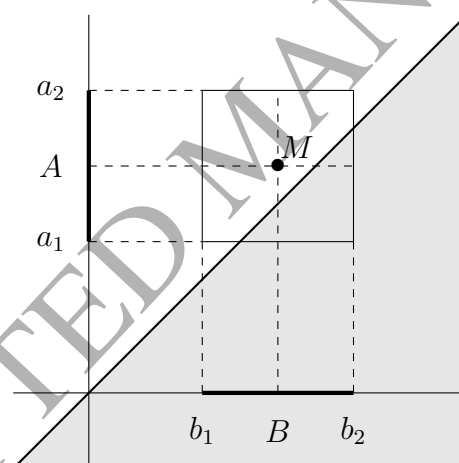

Fig. 2. Graphical interpretation of Midpoint comparison.

Coming back again to the graphical interpretation, in this case $A \leq_M B$ if the center of the rectangle, M, is below the straight line $x = y$ (see Fig. 2).

Clearly this does not define an order, since two different intervals may have the same midpoint. Therefore, in 2006, Z.S. Xu and R.R. Yager ([34]) consider an improved version of this method where they transform it in an order:

Definition 13 *([34]) Let $A = [a_1, a_2]$ y $B = [b_1, b_2]$ two real intervals, then*

$$A \leq_{YX} B \quad \Leftrightarrow \quad a_1 + a_2 < b_1 + b_2 \ or \ (a_1 + a_2 = b_1 + b_2 \ and \ a_2 - a_1 \leq b_2 - b_1).$$

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2.2.3 Lattice order

One of the most widespread methods in the literature is the lattice order. This method is the toughest one but, regardless of the adopted point of view, it can be understand as the most intuitive one.

Definition 14 *([17]) Let $A = [a_1, a_2]$ y $B = [b_1, b_2]$ two real intervals, then*

$$A \leq_{lo} B \quad \Leftrightarrow \quad a_1 \leq b_1 \text{ and } a_2 \leq b_2.$$

Note that this is a partial order. Looking at the graphical representation (see Fig. 3) $A \leq_{lo} B$ if the straight line between the "maxi-min" and "maxi-max" point is below the straight line $x = y$.
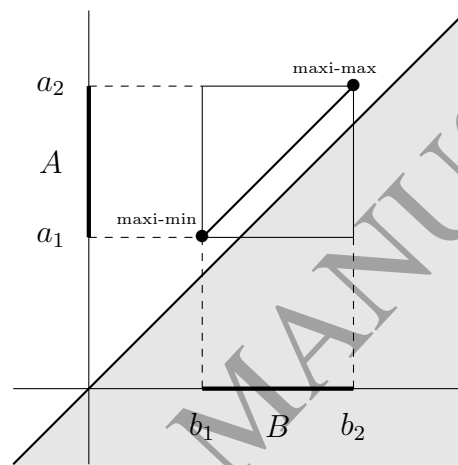

Fig. 3. Graphical interpretation of Lattice order.

Remark 5 *Lattice order coincides with having at the same time the maxi-max dominance and the maxi-min dominance.*

2.2.4 Admissible linear orders

A distinguished family of orders for intervals is the set of admissible orders. This family contains all the linear orders refining the lattice order.

In 2013, Bustince et al. ([8]) generalize some admissible linear orders for intervals using aggregation functions, in general, and the weighted means K_α, in particular. For each $\alpha \in [0, 1]$, the mapping $K_\alpha : [0, 1]^2 \rightarrow [0, 1]$ was defined by Atanassov ([2]):

$$K_\alpha(a, b) = a + \alpha(b - a).$$

The order introduced by Bustince et al. is defined as follows

Definition 15 *([8]) Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two closed real intervals in $L([0,1])$, and two parameters $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then $A \leq_{\alpha,\beta} B$ iff:*

$$K_\alpha(a_1, a_2) < K_\alpha(b_1, b_2) \text{ or } (K_\alpha(a_1, a_2) = K_\alpha(b_1, b_2) \text{ and } K_\beta(a_1, a_2) < K_\beta(b_1, b_2)).$$

Remark 6 *Bustince et al. ([8]) prove that*

- *If $\alpha \in [0,1[$, then all the admissible orders $\leq_{\alpha,\beta}$ with $\beta > \alpha$ coincide. This admissible order is noted as $\leq_{\alpha,+}$.*
- *If $\alpha \in]0,1]$, then all the admissible orders $\leq_{\alpha,\beta}$ with $\beta < \alpha$ coincide. This admissible order is noted as $\leq_{\alpha,-}$.*

Remark 7 *Some of the previously considered linear orders can be recovered as admissible linear orders defined in terms of the K_α operators:*

- *Lexicographical orders with respect to the first ($\leq_{Lex1}$) and the second coordinate ($\leq_{Lex2}$) are recovered by orders $\leq_{\alpha,\beta}$ as the orders $\leq_{0,+}$ and $\leq_{1,-}$, respectively.*
- *Xu and Yager's order $\leq_{YX}$ is recovered by orders $\leq_{\alpha,\beta}$ as the order $\leq_{0.5,+}$.*

By definition, the following remark is trivial.

Remark 8 *Let $\leq_{lo}$ and $\leq_{\alpha,\beta}$ be the orders previously introduced. Then,*

$$\leq_{lo} \implies \leq_{\alpha,\beta} \quad \forall \alpha, \beta \in [0,1].$$

3 Ordering finitely generated sets

The precedent section was a review of several methods used in the literature to order intervals. However, in this section we will try to create an improved version of these methods allowing us to order finitely generated sets.

The main problem we must deal with is the position we adopt with the degenerated intervals in our finitely generated sets. Do they have the same importance as the non-degenerated ones? In this paper we are going to assign the same weight to these two types of intervals. This is due to the fact that focusing in the non-degenerated could derive on "almost surely" orders where the intervals of measure zero do not take part in the order.

An easy example to show our position about the degenerated intervals could be analysing $I = [0.3, 0.4]$ and $J = [0.1, 0.2] \cup \{1\}$. If we consider $I \geq J$, then we do not take into account the degenerated intervals. On the other hand, it is possible to consider there is a doubt in how to order I and J. This is the point of view considered in this paper.

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3.1 α^{sg}-point order

In the previous section we have introduced the mapping K_α which gives us the α-point of an interval. This section is focused on the generalization of this concept to finitely generated sets. The idea is to distribute the interval $[0,1]$ where the parameter α takes its values in as many subintervals as intervals has the finitely generated set (n). Then, each subinterval of $[0,1]$ will be associated to an interval of the finitely generated set and the parameter α will be rescaled to select the corresponding α-point. However, a problem may arise when $\alpha = i/n$ ($i \in \{1, \ldots, n-1\}$), do we select the right end of the first interval or the left end of the second one? Therefore, besides the parameter α, a direction must be fixed too.

This section is divided in two parts, first of all we will introduce the concept of α^{sg}-point and secondly we will use it to define the α^{sg}-point order.

3.1.1 α^{sg}-point of a finitely generated set

As we have said, our aim is to generalize the concept of α-point of an interval to finitely generated sets. However, in this new case, besides the parameter α a direction must be fixed, too. Therefore, our parameter will be in $[0,1] \times \{-, +\}$, but for simplicity we will denote this set by $[0,1]^{\{-,+\}}$.

Definition 16 *Let* $\alpha^{sg1}, \beta^{sg2} \in [0,1]^{\{-,+\}}$. *A linear order on* $[0,1]^{\{-,+\}}$ *is defined as:*

$$\alpha^{sg1} \leq \beta^{sg2} \Leftrightarrow \alpha < \beta \ or \ (\alpha = \beta \ and \ (sg1, sg2) \neq (+, -)),$$

$$\alpha^{sg1} = \beta^{sg2} \Leftrightarrow \alpha = \beta \ and \ sg1 = sg2,$$
$$\alpha^{sg1} < \beta^{sg2} \Leftrightarrow \alpha < \beta \ or \ (\alpha = \beta \ and \ (sg1, sg2) = (-, +)).$$

Proposition 3 *Let* $\leq$ *be the relation introduced in Definition 16. Then,* $\leq$ *is a linear order on* $[0,1]^{\{-,+\}}$.

Proof.

- Reflexivity.

$$\alpha = \alpha \ and \ (sg, sg) \neq (+, -) \ \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}.$$

- Antisymmetry. Given $\alpha^{sg1}, \beta^{sg2} \in [0,1]^{\{-,+\}}$ satisfying

$$\alpha^{sg1} \leq \beta^{sg2} \ and \ \beta^{sg2} \leq \alpha^{sg1}.$$

Then,

$$\alpha = \beta \ and \ (sg1, sg2) \neq (+, -) \neq (sg2, sg1).$$

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Therefore
$$(sg1, sg2) = (-, -) \text{ or } (sg1, sg2) = (+, +).$$

Thus
$$sg1 = sg2 \text{ and } \alpha^{sg1} = \beta^{sg2}.$$

- Transitivity. Given $\alpha^{sg1}, \beta^{sg2}, \gamma^{sg3} \in [0, 1]^{\{-,+\}}$ satisfying

$$\alpha^{sg1} \leq \beta^{sg2} \text{ and } \beta^{sg2} \leq \gamma^{sg3}.$$

Then,

$$\alpha < \gamma \text{ or } \alpha = \beta = \gamma \text{ and } (sg1, sg2) \neq (+, -) \neq (sg2, sg3).$$

Thus,

$$(sg1, sg3) \neq (+, -) \text{ and therefore } \alpha^{sg1} \leq \gamma^{sg3}.$$

- Linear. Given $\alpha^{sg1}, \beta^{sg2} \in [0, 1]^{\{-,+\}}$, as $\alpha, \beta \in [0, 1]$, then $\alpha < \beta$ or $\alpha = \beta$ or $\beta < \alpha$. In the first and third cases we can conclude $\alpha^{sg1} < \beta^{sg2}$ and $\beta^{sg2} < \alpha^{sg1}$, respectively.

  When $\alpha = \beta$, if $(sg1, sg2) = (-, -)$ or $(sg1, sg2) = (+, +)$ then $\alpha^{sg1} = \beta^{sg2}$; if $(sg1, sg2) = (-, +)$ then $\alpha^{sg1} < \beta^{sg2}$, and if $(sg1, sg2) = (+, -)$ then $\beta^{sg2} < \alpha^{sg1}$. Thus, $\leq$ is a linear order. ∎

Remark 9 *Note that this order coincides with the lexicographical order with respect to the first coordinate on $[0, 1] \times C$, where C is the two-element chain $\{-, +\}$ with respecto to the order $- < +$.*

Once the order in $[0, 1]^{\{-,+\}}$ is introduced, some properties which are the basis of this paper are proven.

Proposition 4 *Let $\beta \in [0, 1]$. There does not exist an $\alpha^{sg} \in [0, 1]^{\{-,+\}}$ satisfying $\beta^- < \alpha^{sg} < \beta^+$, where $\beta^-, \beta^+ \in [0, 1]^{\{-,+\}}$.*

Proof.   Assume that there exists $\alpha^{sg} \in [0, 1]^{\{-,+\}}$ satisfying $\beta^- < \alpha^{sg} < \beta^+$, where $\beta^-, \beta^+ \in [0, 1]^{\{-,+\}}$.

$\beta^- < \alpha^{sg}$ implies $\beta \leq \alpha$. On the other hand, $\alpha^{sg} < \beta^+$ implies $\alpha \leq \beta$. Therefore, $\alpha = \beta$.

However, as $\beta^- < \alpha^{sg}$, $sg \neq -$ and, as $\alpha^{sg} \leq \beta^+$, $sg \neq +$. Thus, there does not exist an $\alpha^{sg} \in [0, 1]^{\{-,+\}}$ satisfying this condition. ∎

Once an order in $[0, 1]^{\{-,+\}}$ is defined, we can tackle the study of the finitely generated sets introducing the concept of α^{sg}-point, which is just a generalization of the α-point of an interval.

Before this generalization is given, we need to introduce a map, which will be denoted by Υ. It will allow us to determine the selected interval of the finitely generated set where we will consider the α^{sg}-point.

Definition 17 *Let n be a natural number and $\alpha^{sg} \in [0,1]^{\{-,+\}}$. The function $\Upsilon : [0,1]^{\{-,+\}} \times \mathbb{N} \to \mathbb{N}$ is defined in the following way:*

$$\Upsilon(0^-,n) = \Upsilon(0^+,n) = 1,$$

$$\Upsilon(1^-,n) = \Upsilon(1^+,n) = n,$$

$$\Upsilon(\tfrac{i}{n}^-,n) = i \ \forall i \in \{1,\ldots,n-1\},$$

$$\Upsilon(\tfrac{i}{n}^+,n) = i+1 \ \forall i \in \{1,\ldots,n-1\},$$

$$\Upsilon(\alpha^{sg},n) = i \ \forall \alpha \text{ satisfying } \tfrac{i-1}{n} < \alpha < \tfrac{i}{n} \text{ when } i \in \{1,\ldots,n\}.$$

Definition 18 *Let $A = \bigcup\limits_{i=1}^{n_A} I_i$ be a n_A-finitely generated set and $\alpha^{sg} \in [0,1]^{\{-,+\}}$. The α^{sg}-point of the finitely generated set A is defined as*

$$K_{\alpha^{sg}}(A) = \inf(I_{\Upsilon(\alpha^{sg},n_A)}) + \alpha'\left(\sup(I_{\Upsilon(\alpha^{sg},n_A)}) - \inf(I_{\Upsilon(\alpha^{sg},n_A)})\right),$$

with $\alpha' = n_A \cdot \alpha - \Upsilon(\alpha^{sg},n_A) + 1 \in [0,1]$.

Remark 10 *Values of $\alpha \in \{0,1\}$ always induce the same α^{sg}-points $\forall sg \in \{-,+\}$. Therefore, we consider $0^- = 0^+$ and $1^- = 1^+$.*

The following example shows how these α^{sg}-points are obtained.

Example 1 *Let $A = [0,0.2] \cup [0.3,0.4] \cup [0.7,1] \in FG([0,1))$ and $\alpha^{sg} = 0.25^+$.*

We can easily see that $n_A = 3$ and therefore

$$\Upsilon(\alpha^{sg},3) = \begin{cases} 1 & \text{if } \alpha < \tfrac{1}{3} \text{ or } \alpha^{sg} = \tfrac{1}{3}^-, \\ 2 & \text{if } \tfrac{1}{3} < \alpha < \tfrac{2}{3} \text{ or } \alpha^{sg} \in \{\tfrac{1}{3}^+, \tfrac{2}{3}^-\}, \\ 3 & \text{if } \tfrac{2}{3} < \alpha \text{ or } \alpha^{sg} = \tfrac{2}{3}^+. \end{cases}$$

Moreover, $0 < 0.25 < \tfrac{1}{3}$. Then, $\Upsilon(0.25^+,3) = 1$ and, by Definition 18,

$$\alpha' = n \cdot \alpha - \Upsilon(\alpha^{sg},n) + 1 = 3 \cdot 0.25 - 1 + 1 = 0.75.$$

Thus,

$$K_{\alpha^{sg}}(A) = \inf(I_{\Upsilon(\alpha^{sg},n)}) + \alpha'\left(\sup(I_{\Upsilon(\alpha^{sg},n)}) - \inf(I_{\Upsilon(\alpha^{sg},n)})\right) =$$
$$= \inf(I_1) + 0.75(\sup(I_1) - \inf(I_1)) = 0 + 0.75(0.2 - 0) = 0.15.$$

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Note that when working with intervals, the α^{sg}-point of an interval is exactly the classical α-point and the direction turns unnecessary.

Proposition 5 *Let $A \in L([0,1])$ and $\alpha \in [0,1]$. Then,*

$$K_{\alpha^-}(A) = K_{\alpha^+}(A) = K_\alpha(A).$$

Proof. Let us calculate $K_{\alpha^-}(A)$ and $K_{\alpha^+}(A)$, considering Definition 18:

$$\Upsilon(\alpha^-, 1) = \Upsilon(\alpha^+, 1) = 1,$$

$$\alpha' = n \cdot \alpha - \Upsilon(\alpha^{sg}, n) + 1 = 1 \cdot \alpha - 1 + 1 = \alpha \quad \forall sg \in \{-, +\},$$

$$K_{\alpha^{sg}}(A) = \inf(I_{\Upsilon(\alpha^{sg}, n)}) + \alpha' \left(\sup(I_{\Upsilon(\alpha^{sg}, n)}) - \inf(I_{\Upsilon(\alpha^{sg}, n)})\right) =$$
$$= \inf(I_1) + \alpha(\sup(I_1) - \inf(I_1)) = K_\alpha(A). \blacksquare$$

Proposition 6 *Let $A \in FG([0,1])$, then $K_{\alpha^{sg}}(A) \in A \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$.*

Proof. Consider $A = \bigcup_{i=1}^{n_A} I_i \in FG([0,1])$.

If $\alpha \notin \{0, \frac{1}{n_A}, \ldots, \frac{n_A-1}{n_A}, 1\}$, then $K_{\alpha^{sg}}(A)$ is a strict convex combination of the left and the right ends of one of the intervals $\{I_i\}_{i=1}^{n_A}$. Let suppose, without loss of generality, this interval is I_j. As each strict convex combination of two points is between both, then $K_{\alpha^{sg}}(A) \in I_j \subseteq A$.

If $\alpha \in \{0, \frac{1}{n_A}, \ldots, \frac{n_A-1}{n_A}, 1\}$, then $K_{\alpha^{sg}}(A)$ is one of the bounds of the intervals $\{I_i\}_{i=1}^{n_A}$. As the intervals are closed, then $K_{\alpha^{sg}}(A) \in A$. $\blacksquare$

The following proposition shows how to obtain an α^{sg}-point by rescaling other two α^{sg}-points in the same interval.

Proposition 7 *Let $A \in FG([0,1])$ and let I_A be some of the disjoint intervals which compose A. Let $\alpha^{sg1}, \beta^{sg2}, \gamma^{sg3}$ be three elements in $[0,1]^{\{-,+\}}$ such that $\alpha^{sg1} < \beta^{sg2} < \gamma^{sg3}$ and $K_{\alpha^{sg1}}(A), K_{\beta^{sg2}}(A), K_{\gamma^{sg3}}(A) \in I_A$. Then $K_{\beta^{sg2}}(A) = K_{\alpha^{sg1}}(A) + \dfrac{\beta - \alpha}{\gamma - \alpha}(K_{\gamma^{sg3}}(A) - K_{\alpha^{sg1}}(A))$.*

Proof. Let I_A be the ith interval of A ($i \in \{1, \ldots, n_A\}$).

As $K_{\alpha^{sg1}}(A), K_{\beta^{sg2}}(A), K_{\gamma^{sg3}}(A) \in I_A$, we have:

$$K_{\alpha^{sg1}}(A) = \inf(I_A) + (n_A \cdot \alpha - i + 1)(\sup(I_A) - \inf(I_A)), \tag{1}$$
$$K_{\beta^{sg2}}(A) = \inf(I_A) + (n_A \cdot \beta - i + 1)(\sup(I_A) - \inf(I_A)), \tag{2}$$
$$K_{\gamma^{sg3}}(A) = \inf(I_A) + (n_A \cdot \gamma - i + 1)(\sup(I_A) - \inf(I_A)). \tag{3}$$

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Then, substituting $K_{\alpha^{sg1}}(A)$ and $K_{\gamma^{sg3}}(A)$:

$$K_{\alpha^{sg1}}(A) + \frac{\beta - \alpha}{\gamma - \alpha}(K_{\gamma^{sg3}}(A) - K_{\alpha^{sg1}}(A)) =$$
$$\inf(I_A) + (n_A \cdot \beta - i + 1)(\sup(I_A) - \inf(I_A)) = K_{\beta^{sg2}}(A). \blacksquare$$

As every interval is determined unequivocally by its two ends, it makes sense to study the points that determine unequivocally a finitely generated set. Intuitively, these points are the ends of the intervals forming the finitely generated set. However, these points can be expressed as certain α^{sg}-points: the grid points.

Definition 19 *Let $n \in \mathbb{N}$. A n-grid is defined as:*

$$G(n) = \{\frac{i-1}{n}^+ \mid i = 1, \ldots, n\} \bigcup \{\frac{i}{n}^- \mid i = 1, \ldots, n\}.$$

Taking into account the previous definition, it is necessary to remark a few points.

Remark 11 *Let $n \in \mathbb{N}$ and $G(n)$ a n-grid. Then,*

- $G(n) \subseteq [0,1]^{\{-,+\}}$.
- $\#G(n) = 2n$.

A few important properties of n-grids are shown in the following propositions.

Proposition 8 *Let $n, m \in \mathbb{N}$. Then,*

$$G(n) \subseteq G(m) \quad \Leftrightarrow \quad n|m$$

where $n|m$ represents that n is a divisor of m.

Proof. It is straightforward just considering that $\forall i \in \{1, \ldots, n\}$ there exists $j \in \{1, \ldots, m\}$ such that $\frac{i}{n} = \frac{j}{m}$ and this is equivalent to $n|m$. $\blacksquare$

Proposition 9 *Let $n, m \in \mathbb{N}$. Then,*

$$\frac{1}{n}^{sg} \in G(m) \quad \Leftrightarrow \quad n|m.$$

Proof. If $\frac{1}{n}^{sg} \in G(m)$, then we can express $\frac{1}{n}$ as $\frac{j}{m}$, with $j \in \{1, \ldots, m\}$. Finally, $m = n \cdot j$ and therefore $n|m$.

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If $n|m$, by Proposition 8, $G(n) \subseteq G(m)$. Finally, as $\frac{1}{n}^{sg} \in G(n)$, we conclude $\frac{1}{n}^{sg} \in G(m)$. $\blacksquare$

n-grids are going to be fundamental in the analysis of finitely generated sets. In the following definitions we are going to see a characterization of a finitely generated set using a certain n-grid.

Definition 20 *Let A be a finitely generated set and n a natural number. The set of n-grid points of A is defined as the set of all the α^{sg}-points of A for any $\alpha^{sg} \in G(n)$ and it is denoted by $K^n(A)$. Thus,*

$$K^n(A) = \{K_{\alpha^{sg}}(A) \mid \alpha^{sg} \in G(n)\}.$$

Definition 21 *Let A be a n_A-finitely generated set. The n_A-grid points, $K^{n_A}(A)$, are called fundamental points of A.*

It is immediate that the fundamental points of any set A determine unequivocally a n_A-finitely generated set.

When working with finitely generated sets, some operations between them are often required. In the interval case the basic arithmetic operations can be done just considering the two ends of both intervals. Let us study now how many points are needed in the finitely generated case.

Definition 22 *Let $n, m \in \mathbb{N}$. The n, m-grid is defined as:*

$$G(n, m) = G(n) \cup G(m).$$

Taking into account the previous definition, it is necessary to remark a few points.

Remark 12 *Let $n, m \in \mathbb{N}$ and $G(n, m)$ a n, m-grid. Then,*

- *$G(n, m) \subseteq [0, 1]^{\{-, +\}}$.*
- *$\#G(n, m) = 2(n + m - g.c.d.(n, m))$, where g.c.d. denotes the greatest common divisor.*

Proposition 10 *Let $n_1, n_2, n \in \mathbb{N}$, then*

$$G(n_1, n_2) = G(n) \quad \Leftrightarrow \quad \min(n_1, n_2)|\max(n_1, n_2) \text{ and } \max(n_1, n_2) = n.$$

Proof. Let us suppose $G(n_1, n_2) = G(n_1) \cup G(n_2) = G(n)$. Therefore, $G(n_1) \subseteq G(n)$ and $G(n_2) \subseteq G(n)$. By Proposition 8, $n_1|n$ and $n_2|n$.

In addition, we know that $\frac{1}{n}^+ \in G(n)$. As $G(n) = G(n_1) \cup G(n_2)$, $\frac{1}{n}^+ \in G(n_1)$ or $\frac{1}{n}^+ \in G(n_2)$. Therefore, by Proposition 9, $n|n_1$ or $n|n_2$, i.e. $n_1 = n$ or $n_2 = n$.

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Conversely, let us suppose without loss of generality that $n_1 \leq n_2$, i.e. $\min(n_1, n_2) = n_1$ and $\max(n_1, n_2) = n_2$.

If $n_1 | n_2$ then, by Proposition 8, $G(n_1) \subseteq G(n_2)$ and, as $G(n_1, n_2) = G(n_1) \cup G(n_2)$, then $G(n_1, n_2) = G(n_2) = G(n)$. ∎

An important and intuitive property of the grid points is showed in the following proposition. It demonstrates the continuity of the finitely generated sets between two consecutive grid points with different direction.

Proposition 11 *Let $A, B \in FG([0,1])$, then if β_{i-1}^+, β_i^- are two consecutive elements of $G(n_A, n_B)$ there exist two intervals I_A and I_B satisfying:*

$$K_{\alpha^{sg}}(A) \subseteq I_A \text{ and } K_{\alpha^{sg}}(B) \subseteq I_B, \ \ \forall \alpha^{sg} \in [\beta_{i-1}^+, \beta_i^-].$$

Proof. If $\gamma_{i-1}^+, \gamma_i^-$ are two consecutive elements of $G(n_A)$, by definition of α^{sg}-point there exists an interval I_A satisfying:

$$K_{\alpha^{sg}}(A) \subseteq I_A \ \ \forall \alpha^{sg} \in [\gamma_{i-1}^+, \gamma_i^-]. \tag{4}$$

As $G(n_A) \subseteq G(n_A, n_B)$, if β_{i-1}^+, β_i^- are two consecutive elements of $G(n_A, n_B)$, there exist two consecutive elements of $G(n_A)$, $\gamma_{j-1}^+, \gamma_j^-$, satisfying:

$$[\beta_{i-1}^+, \beta_i^-] \subseteq [\gamma_{j-1}^+, \gamma_j^-]. \tag{5}$$

Finally, joining (4) and (5), if β_{i-1}^+, β_i^- are two consecutive elements of $G(n_A, n_B)$, then:

$$K_{\alpha^{sg}}(A) \subseteq I_A \ \ \forall \alpha^{sg} \in [\beta_{i-1}^+, \beta_i^-].$$

Analogously, we proceed in the same way with B. ∎

Once we have introduced and studied in detail the concept of α^{sg}-points, we will see how to use it to define an order on the class of the finitely generated sets.

3.1.2 α^{sg}-point order

In this subsection we construct an index-based method to order finitely generated sets using the concept of α^{sg}-point.

Definition 23 *Let $A, B \in FG([0,1])$ and a fixed $\alpha^{sg} \in [0,1]^{\{-,+\}}$. Then, a relation on $FG([0,1])$ is defined as:*

$$A \leq_{\alpha^{sg}} B \ \ \Leftrightarrow \ \ K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B),$$

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$$A =_{\alpha^{sg}} B \quad \Leftrightarrow \quad K_{\alpha^{sg}}(A) = K_{\alpha^{sg}}(B),$$
$$A <_{\alpha^{sg}} B \quad \Leftrightarrow \quad K_{\alpha^{sg}}(A) < K_{\alpha^{sg}}(B).$$

Proposition 12 *Let $\leq_{\alpha^{sg}}$ be the relation introduced in Definition 23. Then, $\forall \alpha^{sg} \in [0,1]^{\{-,+\}}$, $\leq_{\alpha^{sg}}$ is an order on $FG([0,1])/ =_{\alpha^{sg}}$, where $FG([0,1])/ =_{\alpha^{sg}}$ denotes the quotient space with respect to the equivalence relation $=_{\alpha^{sg}}$.*

Proof. Let us consider α^{sg} is a fixed point in $[0,1]^{\{-,+\}}$.

- Reflexivity. Let $A \in FG([0,1])$. We have that $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(A)$ and, therefore $A \leq_{\alpha^{sg}} A$.
- Antisymmetry. Let $A, B \in FG([0,1])$. $A \leq_{\alpha^{sg}} B$ and $B \leq_{\alpha^{sg}} A$ imply, respectively, $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B)$ and $K_{\alpha^{sg}}(B) \leq K_{\alpha^{sg}}(A)$. Thus, $K_{\alpha^{sg}}(B) = K_{\alpha^{sg}}(A)$ and therefore $A =_{\alpha^{sg}} B$.
- Transitivity. Let $A, B, C \in FG([0,1])$. $A \leq_{\alpha^{sg}} B$ and $B \leq_{\alpha^{sg}} C$ imply, respectively, $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B)$ and $K_{\alpha^{sg}}(B) \leq K_{\alpha^{sg}}(C)$. Thus, $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(C)$ and therefore $A \leq_{\alpha^{sg}} C$. ∎

In Figure 4 we can see a graphical interpretation of this order. The position of the point $(K_{\alpha^{sg}}(B), K_{\alpha^{sg}}(A))$ below or above the straight line $x = y$ determine, respectively, if $A \leq_{\alpha^{sg}} B$ or $B \leq_{\alpha^{sg}} A$.
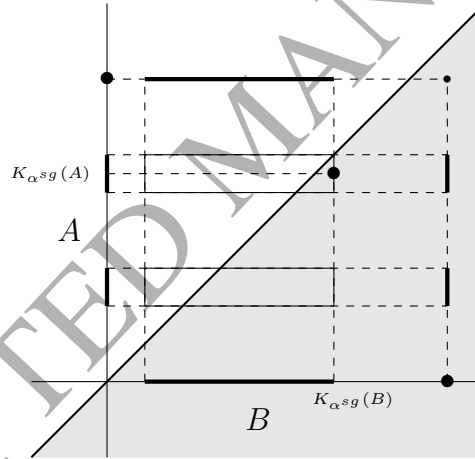

Fig. 4. Graphical interpretation of α^{sg}-point order for finitely generated sets.

Remark 13 *Note that we are defining orders in a quotient space. These orders can also be seen as weak orders in the main space without taking into account the quotient space.*

Remark 14 *This order generalize the midpoint comparison method when restricted to intervals ($\alpha = 0.5$). Similarly, we can generalize the maxi-min dominance with $\alpha = 0$ and the maxi-max dominance with $\alpha = 1$.*

The following subsection is devoted to construct a lattice order for finitely generated sets.

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3.2 Lattice order

We can obtain a lattice order for finitely generated sets that generalizes the lattice order for intervals using the α^{sg}-point.

Definition 24 *Let $A, B \in FG([0,1])$. Then, a relation on $FG([0,1])$ is defined as:*

$$A \leq_{lo} B \quad \Leftrightarrow \quad K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}},$$

$$A =_{lo} B \quad \Leftrightarrow \quad K_{\alpha^{sg}}(A) = K_{\alpha^{sg}}(B) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}},$$

$$A <_{lo} B \quad \Leftrightarrow \quad K_{\alpha^{sg}}(A) < K_{\alpha^{sg}}(B) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}.$$

Proposition 13 *Let $\leq_{lo}$ be the relation introduced in Definition 24. Then, $\leq_{lo}$ is an order on $FG([0,1])$.*

Proof.

- Reflexivity. Let $A \in FG([0,1])$. We have that $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(A) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$ and, therefore $A \leq_{lo} A$.
- Antisymmetry. Let $A, B \in FG([0,1])$. $A \leq_{lo} B$ and $B \leq_{lo} A$ imply, respectively, $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B)$ and $K_{\alpha^{sg}}(B) \leq K_{\alpha^{sg}}(A) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. Thus, $K_{\alpha^{sg}}(B) = K_{\alpha^{sg}}(A) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$ and therefore $A = B$.
- Transitivity. Let $A, B, C \in FG([0,1])$. $A \leq_{lo} B$ and $B \leq_{lo} C$ imply, respectively, $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B)$ and $K_{\alpha^{sg}}(B) \leq K_{\alpha^{sg}}(C) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. Thus, $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(C) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$ and therefore $A \leq_{lo} C$. ∎

Remark 15 *In $(FG([0,1]), \leq_{lo})$ the lowest finitely generated set would be $0 = [0,0]$ and the greatest finitely generated set would be $1 = [1,1]$, due to they are the only finitely generated sets satisfying, respectively, $K_{\alpha^{sg}}([0,0]) = 0$ and $K_{\alpha^{sg}}([1,1]) = 1 \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$.*

In the interval case, the lattice order can be studied considering only the two ends of both intervals. In the same way, in the improved case, the study of every α^{sg}-point of the finitely generated set is not needed and we can reduce the analysis to a certain number of points.

Theorem 1 *Let A and B be two finitely generated sets, then the following statements are equivalent:*

(1) $A \leq_{lo} B$.
(2) $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B) \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$.
(3) $K_{\alpha^{sg}}(A) \leq K_{\alpha^{sg}}(B) \quad \forall \alpha^{sg} \in G(n_A, n_B)$.

Proof. $(i) \Leftrightarrow (ii)$ by definition of lattice order for finitely generated sets.

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$(ii) \Rightarrow (iii)$ is obvious due to $G(n_A, n_B) \subseteq [0,1]^{\{-,+\}}$.

$(iii) \Rightarrow (ii)$ Let $\alpha^{sg} \in [0,1]^{\{-,+\}}$, and β_{i-1}^+, β_i^- the two consecutive elements of $G(n_A, n_B)$ satisfying $\alpha^{sg} \in [\beta_{i-1}^+, \beta_i^-]$.

We have that $K_{\beta_{i-1}^+}(A) \le K_{\beta_{i-1}^+}(B)$ and $K_{\beta_i^-}(A) \le K_{\beta_i^-}(B)$. And we want to prove that $K_{\alpha^{sg}}(A) \le K_{\alpha^{sg}}(B)$.

By Proposition 11, as β_{i-1}^+, β_i^- are two consecutive elements of $G(n_A, n_B)$, then there exists I_A and I_B two intervals satisfying:

$$K_{\beta_{i-1}^+}(A), K_{\beta_i^-}(A) \subseteq I_A.$$

$$K_{\beta_{i-1}^+}(B), K_{\beta_i^-}(B) \subseteq I_B.$$

By Proposition 7, as $\alpha^{sg} \in [\beta_{i-1}^+, \beta_i^-]$ and I_A and I_B are intervals we have:

$$K_{\alpha^{sg}}(A) = K_{\beta_{i-1}^+}(A) + \frac{\alpha - \beta_{i-1}}{\beta_i - \beta_{i-1}} K_{\beta_i^-}(A),$$

$$K_{\alpha^{sg}}(B) = K_{\beta_{i-1}^+}(B) + \frac{\alpha - \beta_{i-1}}{\beta_i - \beta_{i-1}} K_{\beta_i^-}(B).$$

Finally, using $K_{\beta_{i-1}^+}(A) \le K_{\beta_{i-1}^+}(B)$ and $K_{\beta_i^-}(A) \le K_{\beta_i^-}(B)$ in these two equations we arrive to $K_{\alpha^{sg}}(A) \le K_{\alpha^{sg}}(B)$. ■

In Figure 5 we can see a graphical interpretation of this order. By Theorem 1 we can say that $A \le_{lo} B$ (with A and B two finitely generated sets) if and only if $K_{\alpha^{sg}}(A) \le K_{\alpha^{sg}}(B) \quad \forall \alpha^{sg} \in G(n_A, n_B)$, i.e. all the points $(K_{\alpha^{sg}}(B), K_{\alpha^{sg}}(A))$, $\alpha^{sg} \in G(n_A, n_B)$ are below the straight line $y = x$. In the particular case of Figure 5, $A \not\le_{lo} B$ because points 1 and 3 are above the straight line $y = x$. It may be remarked that these two points are respectively related with $\alpha^{sg} = 0^+$ and $\alpha^{sg} = \frac{1}{3}^+$.

It is direct to prove the relation between "lattice order" and "α^{sg}-point order" which is shown in the following proposition.

Proposition 14 *Let $\le_{lo}$ and $\le_{\alpha^{sg}}$ be the orders of finitely generated sets previously introduced.*

$$\le_{lo} \Longrightarrow \le_{\alpha^{sg}} \quad \forall \alpha^{sg} \in [0,1]^{\{-,+\}}.$$

We can observe a toy example of both orders right after.

Example 2 *Let $A = [0, 0.2] \cup [0.3, 0.4] \cup [0.7, 1]$ and $B = [0.3, 0.4]$. Note that either $A, B \in FG[(0,1)]$.*
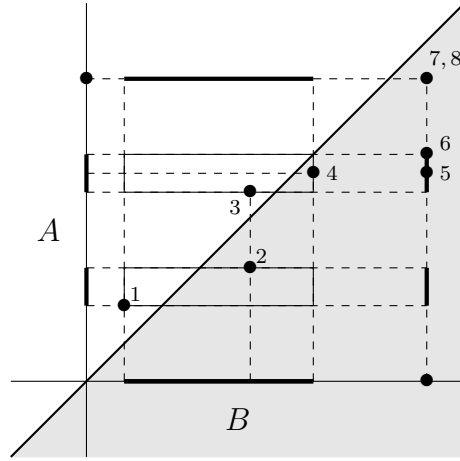
Fig. 5. Graphical interpretation of lattice order for finitely generated sets.

We can easily see that, for instance, when $\alpha^{sg} = 0^+$,

$$K_{0^+}(A) = 0,$$

$$K_{0^+}(B) = 0.3.$$

Therefore, as $K_{0^+}(A) = 0 \leq 0.3 = K_{0^+}(B)$, $A \leq_{0^+} B$.

More generally, for any $\alpha^{sg} \leq 0.5^-$, we can observe that:

$$A \leq_{\alpha^{sg}} B.$$

On the other hand, for any $\alpha^{sg} \geq 0.5^+$, we can observe that:

$$B \leq_{\alpha^{sg}} A.$$

Therefore, we can conclude that $A \not\leq_{lo} B$ and $B \not\leq_{lo} A$.

4 Ordering finite interval-valued hesitant fuzzy sets

In this section we are going to generalize the methods to order finitely generated sets in order to classify finite interval-valued hesitant fuzzy sets. To that end, a fuzzy preference relation needs to be used. Some examples are the proposed by Chen and Lu ([10]), Dubois and Prade ([12]), Kundu ([20]), Nakamura ([22]) or Yuan ([35]). However, it must be remarked that the right choice of a preference relation is not the aim of this paper.

Definition 25 *([16]) Given a finite set of alternatives $\mathcal{A}$, a fuzzy preference relation R is a mapping $R : \mathcal{A} \times \mathcal{A} \to [0, 1]$ such that $R(A, B) + R(B, A) = 1$ for any pair of alternatives A and B in $\mathcal{A}$.*

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This kind of relations are also known in the literature as probabilistic, reciprocal or ipsodual relations, depending on the environment we are working ([6]).

In addition, transitivity for fuzzy preference relations needs to be defined in order to have good properties in this extension. There are many definitions of transitivity between fuzzy preference relations (reader can refer to [31] for further details about fuzzy preference relation transitivity). However, the one which fits better with our problem is the following:

Definition 26 *Let R be a fuzzy preference relation on $\mathcal{A}$. Then R is called consistent iff $\forall(x, y, z) \in \mathcal{A}^3$:*

$$R(x, y) \le R(y, x) \ and \ R(y, z) \le R(z, y) \Rightarrow R(x, z) \le R(z, x).$$

As we have defined the fuzzy preference relations satisfying $R(A, B) + R(B, A) = 1$ for any pair of alternatives A and B, the following corollary is straightforward.

Corollary 1 *Let R be a fuzzy preference relation on $\mathcal{A}$. Then R is consistent iff $\forall(x, y, z) \in \mathcal{A}^3$:*

$$R(x, y) \le 0.5 \ and \ R(y, z) \le 0.5 \Rightarrow R(x, z) \le 0.5.$$

In the following result we can notice that there exists at least one consistent fuzzy preference relation on any family of fuzzy subsets.

Proposition 15 *If the finite set of alternatives $\mathcal{A}$ is formed by fuzzy sets, that is, if $\mathcal{A} \subseteq F(X)$, the family of consistent fuzzy preference relations on $\mathcal{A}$ is not empty.*

Proof. Let $\mathcal{C} : F(X) \to \mathbb{R}$ be the function assigning the middle point of the support for each fuzzy set. The fuzzy preference relation $R(A, B) = \frac{\mathcal{C}(A)}{\mathcal{C}(A) + \mathcal{C}(B)}$ is consistent.

R is a fuzzy preference relation because

$$R(A, B) + R(B, A) = \frac{\mathcal{C}(A)}{\mathcal{C}(A) + \mathcal{C}(B)} + \frac{\mathcal{C}(B)}{\mathcal{C}(A) + \mathcal{C}(B)} = 1.$$

Let us prove that R is consistent. It is straightforward to see that $R(A, B) \le R(B, A)$ is equivalent to $\mathcal{C}(A) \le \mathcal{C}(B)$. Therefore,

$R(A, B) \le R(B, A)$ and $R(B, C) \le R(B, A)$ imply that $\mathcal{C}(A) \le \mathcal{C}(B)$ and $\mathcal{C}(B) \le \mathcal{C}(C)$. Thus, $\mathcal{C}(A) \le \mathcal{C}(C)$ and therefore $R(A, C) \le R(C, A)$, i.e. R is consistent. ∎

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We are going to consider two different families to generalize the orders for finitely generated sets to finite interval-valued hesitant fuzzy sets: "local orders" and "global orders". These two families differ in the way they analyse the α^{sg}-projections: local orders focus on each α^{sg}-projection, but global orders analyse all the α^{sg}-projections together.

4.1 Local orders

In a fixed point $x \in X$, a finite interval-valued hesitant fuzzy set A satisfies that its membership $\mu_A(x)$ is a finitely generated set. Therefore, all these methods seen for ordering finitely generated sets can be extended to finite interval-valued hesitant fuzzy sets. In that case, we are not going to consider the whole membership functions but a fixed α^{sg}-point in each point. This is the concept of α^{sg}-projection which is going to be introduced right after.

Definition 27 *The α^{sg}-projection of a finite interval-valued hesitant fuzzy set A, $P^A_{\alpha^{sg}}$, is defined by the following membership function:*

$$\mu(x) = K_{\alpha^{sg}}(\mu_A(x)) \ \forall x \in X.$$

Remark 16 *Note that each α^{sg}-projection of a finite interval-valued hesitant fuzzy set is a fuzzy set.*

Remark 17 *The set of all α^{sg}-projections of a finite interval-valued hesitant fuzzy set A is set A in itself, i.e.*

$$A = \bigcup_{\alpha^{sg} \in [0,1]^{\{-,+\}}} P^A_{\alpha^{sg}}.$$

Proposition 16 *Let $A, B \in FIVH(X)$, $x \in X$ and $\alpha^{sg} \in [0,1]^{\{-,+\}}$. Then,*

$$P^A_{\alpha^{sg}}(x) \leq P^B_{\alpha^{sg}}(x) \quad \Leftrightarrow \quad \mu_A(x) \leq_{\alpha^{sg}} \mu_B(x).$$

Proof. By definition of α^{sg}-projection, $P^A_{\alpha^{sg}}(x) = K_{\alpha^{sg}}(\mu_A(x))$. On the other hand, by definition of α^{sg}-point order, $\mu_A(x) \leq_{\alpha^{sg}} \mu_B(x)$ if $K_{\alpha^{sg}}(\mu_A(x)) \leq K_{\alpha^{sg}}(\mu_B(x))$. Thus, both conditions are equivalent. ∎

4.1.1 α^{sg}-point order

Once α^{sg}-projections have been introduced, an order between finite interval-valued hesitant fuzzy sets could be determined.

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Definition 28 *Let* $A, B \in FIVH(X)$ *and* R *a consistent fuzzy preference relation. Then, the following relation on* $FIVH(X)$ *could be defined:*

$$A \leq_{\alpha^{sg},R} B \quad \Leftrightarrow \quad R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) \leq 0.5,$$

$$A =_{\alpha^{sg},R} B \quad \Leftrightarrow \quad R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) = 0.5,$$

$$A <_{\alpha^{sg},R} B \quad \Leftrightarrow \quad R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) < 0.5.$$

Proposition 17 *Let* α^{sg} *be in* $[0,1]^{\{-,+\}}$ *and let* R *be a consistent fuzzy preference relation. The relation* $\leq_{\alpha^{sg},R}$ *introduced in Definition 28 is an order on* $FIVH(X)/=_{\alpha^{sg},R}$, *where* $FIVH(X)/=_{\alpha^{sg},R}$ *denotes the quotient space with respect to the equivalence relation* $=_{\alpha^{sg},R}$

Proof.

- Reflexivity. Let $A \in FIVH(X)$. Then, $P_{\alpha^{sg}}^A \in F(X)$ and, by definition of fuzzy preference relation, $R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^A\right) = 0.5$. Therefore, $A \leq_{\alpha^{sg},R} A$.
- Antisymmetry. Let $A, B \in FIVH(X)$. Then, $P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B \in F(X)$. On the other hand, if $A \leq_{\alpha^{sg},R} B$ and $B \leq_{\alpha^{sg},R} A$, then $R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) \leq 0.5$ and $R\left(P_{\alpha^{sg}}^B, P_{\alpha^{sg}}^A\right) \leq 0.5$. Therefore, as $R(A,B) + R(B,A) = 1$, $R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) = 0.5$. Thus, $A =_{\alpha^{sg},R} B$.
- Transitivity. Let $A, B, C \in FIVH(X)$. Then, $P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B, P_{\alpha^{sg}}^C \in F(X)$. On the other hand, if $A \leq_{\alpha^{sg},R} B$ and $B \leq_{\alpha^{sg},R} C$, then $R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) \leq 0.5$ and $R\left(P_{\alpha^{sg}}^B, P_{\alpha^{sg}}^C\right) \leq 0.5$. Then, as R is consistent, $R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^C\right) \leq 0.5$ and therefore, $A \leq_{\alpha^{sg},R} C$. ∎

4.1.2 Lattice order

Along this paper we have analysed different orders for intervals and finitely generated sets and we have already seen the extension of "α^{sg}-points order" to finite interval-valued hesitant fuzzy sets, so it seems intuitive to think about an expansion of "lattice order" to finite interval-valued hesitant fuzzy sets.

Definition 29 *Let* $A, B \in FIVH(X)$ *and* R *a consistent fuzzy preference relation. Then, the following relation on* $FIVH(X)$ *could be defined:*

$$A \leq_{LO,R} B \quad \Leftrightarrow \quad R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) \leq 0.5 \; \forall \alpha^{sg} \in [0,1]^{\{-,+\}},$$

$$A =_{LO,R} B \quad \Leftrightarrow \quad R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) = 0.5 \; \forall \alpha^{sg} \in [0,1]^{\{-,+\}},$$

$$A <_{LO,R} B \quad \Leftrightarrow \quad R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) < 0.5 \; \forall \alpha^{sg} \in [0,1]^{\{-,+\}}.$$

We can see that this relation is actually an order in the following proposition.

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Proposition 18 *Let R be a consistent fuzzy preference relation. The relation $\leq_{LO,R}$ introduced in Definition 29 is an order on $FIVH(X)/ =_{LO,R}$, where $FIVH(X)/ =_{LO,R}$ denotes the quotient space with respect to the equivalence relation $=_{LO,R}$.*

Proof.

- Reflexivity. Let $A \in FIVH(X)$. Then, $P_{\alpha^{sg}}^{A} \in F(X) \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$ and, by definition of fuzzy preference relation, $R\left(P_{\alpha^{sg}}^{A}, P_{\alpha^{sg}}^{A}\right) = 0.5 \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. Therefore, $A \leq_{LO,R} A$.

- Antisymmetry. Let $A, B \in FIVH(X)$. Then, $P_{\alpha^{sg}}^{A}, P_{\alpha^{sg}}^{B} \in F(X) \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. On the other hand, if $A \leq_{\alpha^{sg},R} B$ and $B \leq_{\alpha^{sg},R} A \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$, then $R\left(P_{\alpha^{sg}}^{A}, P_{\alpha^{sg}}^{B}\right) \leq 0.5$ and $R\left(P_{\alpha^{sg}}^{B}, P_{\alpha^{sg}}^{A}\right) \leq 0.5 \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. Therefore, as $R(A,B) + R(B,A) = 1$, $R\left(P_{\alpha^{sg}}^{A}, P_{\alpha^{sg}}^{B}\right) = 0.5$ $\forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. Thus, $A =_{LO,R} B$.

- Transitivity. Let $A, B, C \in FIVH(X)$. Then, $P_{\alpha^{sg}}^{A}, P_{\alpha^{sg}}^{B}, P_{\alpha^{sg}}^{C} \in F(X) \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. On the other hand, if $A \leq_{\alpha^{sg},R} B$ and $B \leq_{\alpha^{sg},R} C \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$, then $R\left(P_{\alpha^{sg}}^{A}, P_{\alpha^{sg}}^{B}\right) \leq 0.5$ and $R\left(P_{\alpha^{sg}}^{B}, P_{\alpha^{sg}}^{C}\right) \leq 0.5 \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$. Then, as R is consistent, $R\left(P_{\alpha^{sg}}^{A}, P_{\alpha^{sg}}^{C}\right) \leq 0.5 \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$ and therefore, $A \leq_{LO,R} C$. ∎

It is direct to compare this order with the previous one in the following proposition.

Proposition 19 *Let $\leq_{LO,R}$ and $\leq_{\alpha^{sg},R}$ be the orders between finite interval-valued hesitant fuzzy sets introduced in Definition 28 and 29, respectively. Then,*

$$\leq_{LO,R} \ \Rightarrow \ \leq_{\alpha^{sg},R} \ \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}.$$

We can observe a toy example illustrating local orders right after.

Example 3 *Let $X = [0,1]$ be the universe where we define $A, B \in FIVH(X)$. Let A be defined by the following membership function:*

$$\mu_A(x) = \begin{cases} 1 & \text{If } x \in [0.2, 1], \\ 0 & \text{If } x \in [0, 0.2). \end{cases}$$

Analogously, let B be defined by the following membership function:

$$\mu_B(x) = \begin{cases} \{0\} \cup \{1\} & \text{If } x \in [0, 0.7), \\ [0.2, 0.8] \cup \{1\} & \text{If } x \in [0.75, 1]. \end{cases}$$

Let $\mathcal{C} : F(X) \to [0,1]$ be the function assigning the middle point of the support

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for each fuzzy set. We will consider the consistent fuzzy preference relation:

$$R(U,V) = \frac{\mathcal{C}(U)}{\mathcal{C}(U) + \mathcal{C}(V)}.$$

We can observe that A is a fuzzy set and, therefore, for any $\alpha^{sg} \in [0,1]^{\{-,+\}}$,

$$P_{\alpha^{sg}}^A = A.$$

On the other hand, the family of $P_{\alpha^{sg}}^B$ is defined by the following membership functions:

For any $\alpha^{sg} \leq 0.5^-$, $\quad \mu_{P_{\alpha^{sg}}^B}(x) = \begin{cases} 0 & \text{If } x \in [0, 0.7), \\ 0.2 + 1.2\alpha & \text{If } x \in [0.7, 1]. \end{cases}$

For any $\alpha^{sg} \geq 0.5^+$, $\quad \mu_{P_{\alpha^{sg}}^B}(x) = 1.$

Therefore,

For any $\alpha^{sg} \leq 0.5^-$, $\quad \mathcal{C}(P_{\alpha^{sg}}^B) = 0.85$ and $\mathcal{C}(P_{\alpha^{sg}}^A) = \mathcal{C}(A) = 0.6.$

For any $\alpha^{sg} \geq 0.5^+$, $\quad \mathcal{C}(P_{\alpha^{sg}}^B) = 0.5$ and $\mathcal{C}(P_{\alpha^{sg}}^A) = \mathcal{C}(A) = 0.6.$

Finally, we can see that, if $\alpha^{sg} \leq 0.5^-$,

$$R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) = \frac{\mathcal{C}(P_{\alpha^{sg}}^A)}{\mathcal{C}(P_{\alpha^{sg}}^A) + \mathcal{C}(P_{\alpha^{sg}}^B)} = \frac{0.6}{0.6 + 0.85} \leq 0.5$$

Therefore,

$$A \leq_{\alpha^{sg}, R} B.$$

Analogously, we can see that, if $\alpha^{sg} \geq 0.5^-$,

$$R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) = \frac{\mathcal{C}(P_{\alpha^{sg}}^A)}{\mathcal{C}(P_{\alpha^{sg}}^A) + \mathcal{C}(P_{\alpha^{sg}}^B)} = \frac{0.6}{0.6 + 0.5} \geq 0.5$$

Therefore,

$$A \nleq_{\alpha^{sg}, R} B.$$

We can conclude that, as $A \leq_{\alpha^{sg}, R} B.$ is not hold for every α^{sg},

$$A \nleq_{LO, R} B.$$

4.2   Global orders

Finally, we are going to introduce global orders. This kind of orders are characterized by their global analysis of all the α^{sg}-projections. Instead of comparing

the α^{sg}-projections of the pair of finite interval-valued hesitant fuzzy sets for any α^{sg}, they obtain a finite interval-valued hesitant fuzzy preference relation for each finite interval-valued hesitant fuzzy set and then they compare both relations.

4.2.1 Finite interval-valued hesitant fuzzy preference relation order

Let us introduce finite interval-valued hesitant fuzzy preference relations, which are going to allow us to obtain a new order between finite interval-valued hesitant fuzzy sets.

Definition 30 *Given a finite set of alternatives $\mathcal{A}$, a finite interval-valued hesitant fuzzy preference relation R is a mapping $R : \mathcal{A} \times \mathcal{A} \to FG([0,1])$ such that $R(A, B)$ is symmetric to $R(B, A)$ in relation to the point 0.5 for any pair of alternatives A and B in $\mathcal{A}$.*

Remark 18 *Note that $R(A, B)$ and $R(B, A)$ are finitely generated sets. Therefore, being symmetric in relation to the point 0.5 would be that $K_{\alpha^{sg}}(R(A, B)) + K_{(1-\alpha)^{sg^c}}(R(B, A)) = 1 \ \forall \alpha^{sg} \in [0,1]^{\{-,+\}}$.*

Remark 19 *In [11], Chen et al. introduced interval-valued hesitant fuzzy preference relation. However, they did not consider finiteness, which is absolutely necessary in order to consider α^{sg}-points. In addition, in their definition they did not consider finite unions of disjoint closed intervals but finite unions of closed intervals. The absence of disjointedness could seem insignificant, but makes both definitions really different when dealing with them. Furthermore, considering α^{sg}-points do not make sense with their proposal. For instance, $R(A, B) = \{[0.3, 0.6], [0.4, 0.7]\}$ would not be a finite interval-valued hesitant fuzzy preference relation in our sense, but it would be a (finite) interval-valued hesitant fuzzy preference relation in Chen et al.'s sense.*

In addition, a way of transitivity between finite interval-valued hesitant fuzzy preference relations must be established. We have generalized the consistency between fuzzy preference relations in the following definition.

Definition 31 *Let R be a finite interval-valued hesitant fuzzy preference relation on $\mathcal{A}$ and $\leq_*$ an order between finitely generated sets. Then, R is called $\leq_*$-consistent iff $\forall (x, y, z) \in \mathcal{A}^3$:*

$$R(x, y) \leq_* R(y, x) \ and \ R(y, z) \leq_* R(z, y) \Rightarrow R(x, z) \leq_* R(z, x).$$

From the following proposition we can notice that the family of $\leq_*$-consistent finite interval-valued hesitant fuzzy preference relations is not empty.

Proposition 20 *If the finite set of alternatives $\mathcal{A}$ is formed by finite interval-*

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valued hesitant fuzzy sets, that is, if $\mathcal{A} \subseteq FIVH(X)$, the family of $\leq_*$-consistent finite interval-valued hesitant fuzzy preference relations on $\mathcal{A}$ is not empty.

Proof. Let $\mathcal{C} : FIVH(X) \rightarrow \mathbb{R}$ be the function assigning the middle point of the support for each fuzzy set. The following finite interval-valued hesitant fuzzy preference relation is $\leq_{lo}$-consistent:

$$R(A, B) = \begin{cases} \left[\frac{\mathcal{C}(A)}{\mathcal{C}(A)+\mathcal{C}(B)}, 1\right] & \text{If } \mathcal{C}(A) > \mathcal{C}(B) \\ \left[\frac{1}{4}, \frac{3}{4}\right] & \text{If } \mathcal{C}(A) = \mathcal{C}(B) \\ \left[0, \frac{\mathcal{C}(A)}{\mathcal{C}(A)+\mathcal{C}(B)}\right] & \text{If } \mathcal{C}(A) < \mathcal{C}(B) \end{cases}.$$

R is a finite interval-valued hesitant fuzzy preference relation because, by construction, $R(A, B) = 1 - R(B, A)$, i.e. $R(A, B)$ is symmetric to $R(B, A)$ in relation to the point 0.5.

Let us prove that R is $\leq_{lo}$-consistent. Firstly, we need to prove that $R(A, B) \leq_{lo} R(B, A)$ is equivalent to $\mathcal{C}(A) \leq \mathcal{C}(B)$. This is straightforward considering that we have three possibilities for $R(A, B)$ and $R(B, A)$ ($a \in [0, 0.5)$):

(1) $R(A, B) = [0, a]$ and $R(B, A) = [1 - a, 1]$.
(2) $R(A, B) = [0.25, 0.75]$ and $R(B, A) = [0.25, 0.75]$.
(3) $R(A, B) = [1 - a, 1]$ and $R(B, A) = [0, a]$.

In the first two cases $R(A, B) \leq_{lo} R(B, A)$ and $\mathcal{C}(A) \leq \mathcal{C}(B)$ are satisfied and in the third case none of both is fulfilled. Therefore, $R(A, B) \leq_{lo} R(B, A)$ is equivalent to $\mathcal{C}(A) \leq \mathcal{C}(B)$.

Finally, $R(A, B) \leq_{lo} R(B, A)$ and $R(B, C) \leq_{lo} R(B, A)$ imply that $\mathcal{C}(A) \leq \mathcal{C}(B)$ and $\mathcal{C}(B) \leq \mathcal{C}(C)$. Thus, $\mathcal{C}(A) \leq \mathcal{C}(C)$ and therefore $R(A, C) \leq_{lo} R(C, A)$, i.e. R is $\leq_{lo}$-consistent. ∎

Once we have defined these two concepts a new order between finite interval-valued hesitant fuzzy sets could be introduced.

Definition 32 *Let $A, B \in FIVH(X)$, $\leq_*$ an order between finitely generated sets and R a $\leq_*$-consistent finite interval-valued hesitant fuzzy preference relation. Then, the following relation on $FIVH(X)$ could be defined:*

$$A \leq_{*,R} B \quad \Leftrightarrow \quad R(A, B) \leq_* R(B, A),$$

$$A =_{*,R} B \quad \Leftrightarrow \quad R(A, B) =_* R(B, A),$$

$$A <_{*,R} B \quad \Leftrightarrow \quad R(A, B) <_* R(B, A).$$

Proposition 21 *Let $\leq_*$ be an order between finitely generated sets and let R be a $\leq_*$-consistent finite interval-valued hesitant fuzzy preference relation. The relation $\leq_{*,R}$ introduced in Definition 32 is an order on $FIVH(X)/=_{*,R}$, where $FIVH(X)/=_{*,R}$ denotes the quotient space with respect to the equivalence relation $=_{*,R}$.*

Proof.

- Reflexivity. Let $A \in FIVH(X)$. Then, by definition of finite interval-valued hesitant fuzzy preference relation, $R(A,A) \in FG([0,1])$. Therefore, $R(A,A) \leq_* R(A,A)$. Thus, $A \leq_{*,R} A$.
- Antisymmetry. Let $A, B \in FIVH(X)$. Then, by definition of finite interval-valued hesitant fuzzy preference relation, $R(A,B), R(B,A) \in FG([0,1])$. If $A \leq_{*,R} B$ and $B \leq_{*,R} A$, then $R(A,B) \leq_* R(B,A)$ and $R(B,A) \leq_* R(A,B)$. As $\leq_*$ is an order between finitely generated sets, then $R(A,B) =_* R(B,A)$ and therefore, $A =_{*,R} B$.
- Transitivity. Let $A, B, C \in FIVH(X)$. Then, by definition of finite interval-valued hesitant fuzzy preference relation, $R(A,B), R(B,C) \in FG([0,1])$. If $A \leq_{*,R} B$ and $B \leq_{*,R} C$, then $R(A,B) \leq_* R(B,A)$ and $R(B,C) \leq_* R(C,B)$. As R is a $\leq_*$-consistent hesitant fuzzy preference relation, then $R(A,B) \leq_* R(B,C)$ and therefore, $A \leq_{*,R} C$. ∎

In the following we present a toy example illustrating global orders.

Example 4 *Let $X = [0,1]$ be the universe where we define $A, B \in FIVH(X)$. Let A be defined by the following membership function:*

$$\mu_A(x) = \begin{cases} 1 & If\ x \in [0.2, 1], \\ 0 & If\ x \in [0, 0.2). \end{cases}$$

Analogously, let B be defined by the following membership function:

$$\mu_B(x) = \begin{cases} \{0\} \cup \{1\} & If\ x \in [0, 0.7), \\ [0.2, 0.8] \cup \{1\} & If\ x \in [0.75, 1]. \end{cases}$$

Let $\mathcal{C} : F(X) \to [0,1]$ be the function assigning the middle point of the support for each fuzzy set. We will consider the order on finitely generated sets $\leq_{lo}$ and the $\leq_{lo}$-consistent fuzzy preference relation:

$$R(U,V) = \begin{cases} \left[\frac{\mathcal{C}(U)}{\mathcal{C}(U)+\mathcal{C}(V)}, 1\right] & If\ \mathcal{C}(U) > \mathcal{C}(V) \\ \left[\frac{1}{4}, \frac{3}{4}\right] & If\ \mathcal{C}(U) = \mathcal{C}(V) \\ \left[0, \frac{\mathcal{C}(U)}{\mathcal{C}(U)+\mathcal{C}(V)}\right] & If\ \mathcal{C}(U) < \mathcal{C}(V) \end{cases}.$$

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Therefore,

$$\mathcal{C}(A) = 0.6 \ and \ \mathcal{C}(B) = 0.5,$$
$$R(A, B) = [\frac{0.6}{1.1}, 1] \ and \ R(B, A) = [0, \frac{0.5}{1.1}].$$

We can easily observe that:

$$R(B, A) \leq_{lo} R(A, B).$$

Then,

$$B \leq_{lo,R} A.$$

4.2.2 Finite interval-valued hesitant fuzzy preference relations generated by fuzzy preference relations

We have seen how to construct an order between finite interval-valued hesitant fuzzy sets using finite interval-valued hesitant fuzzy preference relations. However, a fundamental question may arise at this moment, could we generate a finite interval-valued hesitant fuzzy preference relation by a fuzzy preference relation? Under which conditions?

Definition 33 *The class of bounded finite interval-valued hesitant fuzzy sets over X is defined as:*

$$FIVHB(X) = \{A \in TH(X) | \exists n \in \mathbb{N} \ s.t. \ \forall x \in X, \exists m \leq n \ s.t. \ \mu_A(x) \in FG_m(X)\}.$$

Definition 34 *A fuzzy preference relation R is said to be parametrically continuous if $\forall A, B \in IVFS(X)$, $f : [0,1] \to [0,1]$ defined via $f_{A,B}(\alpha) = R\left(P_{\alpha^+}^A, P_{\alpha^+}^B\right)$ is continuous.*

These definitions allow us to establish some conditions under which a fuzzy preference relation generates a finite interval-valued hesitant fuzzy preference relation.

Theorem 2 *Let R be a parametrically continuous fuzzy preference relation and a finite set of alternatives $\mathcal{A} \subset FIVHB(X)$, then $\sigma(R)$ is a finite interval-valued hesitant fuzzy preference relation on $\mathcal{A}$ where*

$$\sigma(R)(A, B) = \bigcup_{\alpha^{sg}} R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right),$$

for any $A, B \in FIVHB(X)$.

Proof. Firstly, we need to prove that $\sigma(R)(A, B)$ is well defined, i.e. $\sigma(R)(A, B) \in FG([0,1]) \ \forall A, B \in FIVHB(X)$.

Let $A, B \in FIVHB(X)$. Then, it exists n_{max}, the maximum natural number such that $\mu_A(x) \in FG_{n_{max}}([0,1])$ or $\mu_B(x) \in FG_{n_{max}}([0,1])$ for some $x \in X$.

Let $G = \bigcup\limits_{i=1}^{n_{max}} G(i)$. Then, for any $\alpha_{i-1}^+, \alpha_i^-$ two consecutive elements of G, there does not exist a discontinuity between $P_{\alpha_{i-1}^-}^A(x)$ and $P_{\alpha_i^+}^A(x)$ or between $P_{\alpha_{i-1}^-}^B(x)$ and $P_{\alpha_i^+}^B(x)$ for any $x \in X$.

Note that $\bigcup\limits_{\alpha_{i-1}^+ \leq \alpha^{sg} \leq \alpha_i^-} P_{\alpha^{sg}}^A$ and $\bigcup\limits_{\alpha_{i-1}^+ \leq \alpha^{sg} \leq \alpha_i^-} P_{\alpha^{sg}}^A$ are interval-valued fuzzy sets. Thus, as R is parametrically continuous, $\bigcup\limits_{\alpha_{i-1}^+ \leq \alpha^{sg} \leq \alpha_i^-} R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right)$, is a closed interval (applying Weierstrass's Theorem and Intermediate Value Theorem to $f_{A,B}(\alpha) = R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right)$).

Finally, as $\#G$ is finite, $\sigma(R)(A, B)$ is a finite union of closed intervals, i.e. $\sigma(R)(A, B) \in FG([0, 1])$.

On the other hand, we need to prove that $\sigma(R)(A, B)$ is symmetric to $\sigma(R)(B, A)$ in relation to the point 0.5. This is straightforward because R is a fuzzy preference relation and then, $R\left(P_{\alpha^{sg}}^A, P_{\alpha^{sg}}^B\right) = 1 - R\left(P_{\alpha^{sg}}^B, P_{\alpha^{sg}}^A\right) \forall \alpha^{sg} \in [0, 1]^{\{-,+\}}$.
∎

5 Conclusions and future research

In this paper several methods to order intervals have been reviewed and all these methods have been generalized to finitely generated sets, which are a finite union of disjoint closed intervals. We have particularized in two orders: "lattice order" and "α^{sg}-point order".

Furthermore, we have seen that the membership function of a finite interval-valued hesitant fuzzy set is a finitely generated set, so we have developed methods to order finite interval-valued hesitant fuzzy sets using orders defined between finitely generated sets. This work has led us to construct, using preference relations, two families of orders between finite interval-valued hesitant fuzzy sets: "local orders" and "global orders". These two families of orders between finite interval-valued hesitant fuzzy sets focus on two different problems and can elect different winners when applied to the same problem. However, in most ordering problems there is not a universal truth and different orders can naturally elect different winners. Furthermore, finite interval-valued hesitant fuzzy preference relations have been introduced. In addition, some conditions allowing a fuzzy preference relation to generate a finite interval-valued hesitant fuzzy preference relation were established.

In the future we will intend to model a group decision making problem with fi-

nite interval-valued hesitant fuzzy sets. In particular, we would like to continue with the real problem of Human Reliability started in [23].

Acknowledgements

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