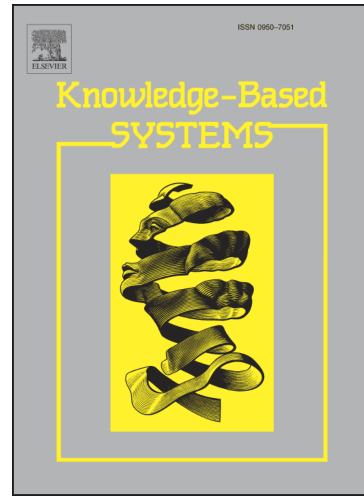

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An entropy measure definition for finite interval-valued hesitant fuzzy sets

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Abstract

In this work , a de nition of entropy is studied in an interval-valued hesitant fuzzy environment , instead of the classical fuzzy logic or the interval-valued one. As the properties of this kind of sets are more complex, the entropy is built by three different functions, where each one represents a different measure: fuzziness, lack of knowledge and hesitance. Using all, an entropy measure for interval-valued hesitant fuzzy sets is obtained, quantifying various types of uncertainty.

From this definition, several results have been developed for each mapping that shapes the entropy measure in order to get such functions with ease, and as a consequence, allowing to obtain this new entropy in a simpler way.

Key words: fuzzy sets, hesitant fuzzy sets, interval-valued hesitant fuzzy sets, entropy, fuzziness, lack of knowledge, hesitance.

1 Introduction

The fuzzy logic was introduced by Zadeh in 1971 (see [32]), becoming a generalization of the classical set theory which has been widely studied since then up to now. The goal of this approach is to represent certain properties that are not possible to be dealt with by the classical logic. It has been applied to a wide range of topics, such as protection of privacy (see [21,22]) or image processing (see [2,23]).

A fuzzy set is characterized by a membership function that depends on the expert that shapes it. In order to overcome this problem, generalizations of the fuzzy sets were carried out. In particular, the interval-valued fuzzy sets were introduced by Sambuc in 1975 (see [25]), where the membership function gives for each point not a single value but an interval. Atanassov's intuitionistic fuzzy sets, developed by Atanassov in 1986 (see [1]), are another generalization where a set is associated to both a membership and a non-membership function. A greater extension of the fuzzy sets are the 2-type fuzzy sets, given by Zadeh in 1975 (see [33]), where for each point, the membership function is defined over the referential $[0, 1]$.

However, type-2 fuzzy sets are difficult to work with, so in 2009 hesitant sets were introduced by Torra (see [27,28]) as an intermediate kind of fuzzy sets. The membership function of a hesitant set assigns a subset of the closed interval $[0, 1]$ instead of a fuzzy set to each point. This property makes them more manageable than type-2 fuzzy sets. In fact, these sets were already introduced by Grattan-Guinness [15] in 1976, with the name of set-valued fuzzy sets. However, Torra provided functional definitions of union and intersection for such sets which were not considered by Grattan-Guinness. This type of sets is currently a rising researching topic, due to the possibilities that they provide (see [5,14,30]), and specially, in decision making (see [10,16,29]). Different extensions of this hesitant sets have been developed lately (see [24]). In our paper, the used and studied generalization is the finite interval-valued hesitant fuzzy sets, given by Pérez *et al.* in 2014 (see [20]).

The study of entropy measures in the fuzzy set theory also became an important part of the research, firstly defined by De Luca and Termini in 1972 (see [11]), whose aim is to quantify the uncertainty associated to a fuzzy set. This concept has been adapted to other types of fuzzy sets, such as Atanassov's intuitionistic fuzzy sets (see [18]), interval-valued fuzzy sets (see [6]) or even interval-valued hesitant fuzzy sets (see [14]).

Nevertheless, the existing definition of entropy for interval-valued hesitant fuzzy sets in [14] only reflects one type of uncertainty, associated to how distant a set is from a union of crisp sets. Our proposal along the work is to define a new entropy measure for interval-valued hesitant fuzzy sets, where three types of uncertainty are reflected through three mappings, instead of the classical concept of just one function for one type of uncertainty associated. In addition, several results have been developed in order to obtain such mappings with ease, and as a result, the entropy measure can be obtained with simpler conditions. Note that this has also been the approach in [18] for the Atanassov intuitionistic fuzzy setting.

The remainder of the paper is structured as follows: the following section is split into three subsections with preliminary concepts about fuzzy sets, hesitant fuzzy sets and entropy and dissimilarity measures respectively. Section 3 details the study related to the new definition of entropy in a interval-valued hesitant environment. In Section 4 the main conclusions of this work are highlighted.

2 Preliminaries

Necessary concepts to understand the definition of entropy proposed in this work are given in this section. It has been split into three subsections. General basic concepts about the fuzzy logic are explained in the former. The used generalization of fuzzy sets, the hesitant fuzzy sets, are developed in the second one. In the latter, the definitions of entropy and dissimilarity measure in different environments are given.

2.1 Fuzzy sets and its extensions

The concepts about the usual types of fuzzy sets can be found in a wide range of sources, such as [9]. These types of sets are important in order to understand the utility provided by the hesitant fuzzy sets, starting with the definition of the classic fuzzy set, which was given for the first time by Zadeh (see [32]).

Definition 1 *Let X be a non-empty set. Given the membership function:*

$$\mu_A : X \to [0, 1],$$

then, the set $A = \{(x, \mu_A(x)) | x \in X\}$ is a fuzzy set in X.

Given $x \in X$, the value $\mu_A(x)$ is called membership degree of x.

Remark 2 *$FS(X)$ denotes the set of all fuzzy sets in X.*

In addition to the definition of fuzzy set, the following concepts are introduced in order to develop the forthcoming results.

Definition 3 *Given $A, B \in FS(X)$, with their membership functions μ_A and μ_B respectively:*

- *The complement of A with respect to the standard negation, which is denoted by A^c, is the fuzzy set given by $A^c = \{(x, 1 - \mu_A(x)) | x \in X\}$.*
- *The partial ordering relation used for fuzzy sets is given by:*

$$A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \ \forall x \in X.$$

- *The set $\xi \in FS(X)$ is called equilibrium set if it is defined as $\xi = \{(x, 0.5) | x \in X\}$.*

The interval-valued fuzzy sets are a generalization of the fuzzy sets, where an interval instead of just one value is associated to each point. This kind of sets were developed by Sambuc (see [25]).

Definition 4 *Let X be a non-empty set. Given the membership function:*

$$\mu_A : X \to L([0, 1]),$$

where $L([0, 1])$ denotes the family of all closed subintervals of $[0, 1]$, then, the set $A = \{(x, \mu_A(x) = [\mu_A(x)^L, \mu_A(x)^U]) | x \in X\}$ is an interval-valued fuzzy set in X.

Remark 5 *$IVFS(X)$ denotes the set of all interval-valued fuzzy sets in X.*

Some useful concepts are introduced in the following definition.

Definition 6 *Given $A, B \in IVFS(X)$, with their membership functions μ_A and μ_B respectively,*

- *The complement of A with respect to the standard negation, which is denoted by A^c, is given by $A^c = \{(x, \mu_{A^c}(x)) | x \in X\}$, where $\mu_{A^c}(x) = [1 - \mu_A(x)^U, 1 - \mu_A(x)^L], \ \forall x \in X$,*
- *The partial ordering relation used in our paper for interval-valued fuzzy sets, is well known and can be found in several sources such as [3,19]. It is given by:*

$$A \leq B \Leftrightarrow \mu_A(x) \leq_I \mu_B(x), \ \forall x \in X,$$

  where $\forall x \in X$

$$\mu_A(x) \leq_I \mu_B(x) \Leftrightarrow \mu_A(x)^L \leq \mu_B(x)^L \text{ and } \mu_A(x)^U \leq \mu_B(x)^U,$$

- *The set $A = \{(x, [0, 1]) | x \in X\}$ is called the pure interval-valued fuzzy set.*

The concept of pure interval-valued fuzzy set is obtained directly from the concept of pure Atanassov intuitionistic fuzzy set introduced in [18], taking into account the mathematical duality between both concepts (see [26]).

In addition, type-2 fuzzy sets were also developed by Zadeh. They represent a generalization of the classical notion of fuzzy set (see [33]).

Definition 7 *Let X be a non-empty set. Given the membership function:*

$$\mu_A : X \to FS([0,1]),$$

then, $A = \{(x, \mu_A(x)) | x \in X\}$ is a type-2 fuzzy set in X.

Remark 8 *$T2FS(X)$ denotes the set of all type-2 fuzzy sets in X. As we will work on a subset of $T2FS(X)$, we are not going to comment any operation on type-2 fuzzy sets, in order to avoid unnecessary explanations.*

In the next subsection, basic concepts about hesitant fuzzy sets are studied. This type of sets represents an intermediate step between the interval-valued fuzzy sets and the 2-type fuzzy sets, which makes them interesting to study and work with. The reason lies in the fact that the type-2 fuzzy sets are hard to handle and use, while the hesitant fuzzy sets have properties that make them more manageable. Furthermore, all the results obtained in a hesitant environment can be quickly adapted to other types of sets, such as interval-valued fuzzy sets and the classical fuzzy sets, since they are a generalization of them.

2.2 Hesitant fuzzy sets

Hesitant fuzzy logic, recently defined by Torra in [27,28], was first introduced by Grattan-Guinnes in [15], with the name of set-valued fuzzy set. Another related developments were carried out in other papers such as [5], where basic definitions about this topic can be found.

Let $\mathcal{P}([0,1])$ denote the family of subsets of the closed interval $[0,1]$. A typical hesitant fuzzy set is defined as follows (see [4,5]):

Definition 9 *Let X be a non-empty set and $\mathbb{H} \subset \mathcal{P}([0,1])$ the set of all finite non-empty subsets of the interval $[0,1]$. Given the membership function:*

$$\mu_A : X \to \mathbb{H},$$

then, the set $A = \{(x, \mu_A(x)) | x \in X\}$ is a typical hesitant fuzzy set in X.

Remark 10 *$THFS(X)$ denotes the set of all typical hesitant fuzzy sets in X.*

As the previous definition states, the membership function of a typical hesitant fuzzy set provides for each element of X a finite subset of the interval $[0,1]$. In order to be able to apply this type of sets in practise, it is desirable to replace finite subsets by subsets which are generated by a union of a finite number of closed intervals. This reasoning leads to a new definition of hesitant fuzzy sets, the finite interval-valued hesitant fuzzy sets, given by Pérez *et al.* (see [20]).

Before providing the definition of finite interval-valued hesitant fuzzy sets, the notions of finitely generated set as well as the complement of such sets are introduced.

Definition 11 *Let $n \in \mathbb{N}$. The class of n-finitely generated sets in $[0,1]$ is given by:*

$$FG_n([0,1]) = \{I \subseteq [0,1] | I = \bigcup_{i=1}^{n} I_i \text{ with } I_i \cap I_j = \emptyset, \forall i \neq j\},$$

where I_i denotes a closed interval in $[0,1]$, for any $i \in \{1,\ldots,n\}$. The class of finitely generated sets in $[0,1]$ is given by:

$$FG([0,1]) = \bigcup_{n=1}^{\infty} FG_n([0,1]).$$

Definition 12 *Let $I = I_1 \cup \cdots \cup I_n$ be an element of $FG_n([0,1])$, where for every $i = 1,\ldots,n$, $I_i = [I_i^L, I_i^U]$. Then, the complement of I is defined as $I^c = I_1^c \cup \cdots \cup I_n^c$, with $I_i^c = [1 - I_i^U, 1 - I_i^L]$, for $i = 1,\ldots,n$.*

Remark 13 *Note that $I \in FG_n([0,1]) \Leftrightarrow I^c \in FG_n([0,1])$.*

After these prior concepts, the definition of an interval-valued hesitant fuzzy set is given as follows (see [20]).

Definition 14 *Let X be a non-empty set. Given the membership function:*

$$\mu_A : X \to FG([0,1]),$$

then, the set $A = \{(x, \mu_A(x)) | x \in X\}$ is an interval-valued hesitant fuzzy set in X.

Remark 15 *$IVHFS(X)$ denotes the set of all interval-valued hesitant fuzzy sets in X.*

The notion of complement is now introduced. This concept is based on the definition of complement of finitely generated sets (Definition 12).

Definition 16 *Let A be an interval-valued hesitant fuzzy set in X with $A = \{(x, \mu_A(x))|x \in X\}$, the complement of A is defined as $A^c = \{(x, \mu_{A^c}(x))|x \in X\}$, where $\mu_{A^c}(x)$ is the complement of a finitely generated defined accordingly to Definition 12.*

For every interval-valued hesitant fuzzy set, and for each point $x \in X$, the membership function belongs to $FG_{n_x}([0,1])$ for some $n_x \in \mathbb{N}$, which represents the number of disjoint closed subintervals that generate the finitely generated set. Obviously, in a interval-valued hesitant fuzzy set some of the closed subintervals can be degenerated, i.e., singletons. If all the intervals are degenerated, then we recover typical hesitant fuzzy sets.

Regarding how to compare two interval-valued hesitant fuzzy sets by an ordering relation, Pérez *et al.* in [20] developed a methodology based on the notion of α^{sg}-points. The one that has been used in this paper is closely related to Xu and Yager (see [31]) total ordering relation for intervals.

Definition 17 *Let $\mathbf{x} = [x^L, x^U], \mathbf{y} = [y^L, y^U] \in L([0,1])$. If the score function is defined by $S(\mathbf{x}) = x^U - x^L$ and the accuracy function by $H(\mathbf{x}) = x^L + x^U$, then the total ordering relation $\leq_{XY}$ is given as follows:*

$$\mathbf{x} \leq_{XY} \mathbf{y} \Longleftrightarrow \begin{cases} H(\mathbf{x}) < H(\mathbf{y}), \\ or \\ H(\mathbf{x}) = H(\mathbf{y}) \quad and \quad S(\mathbf{x}) \leq S(\mathbf{y}). \end{cases}$$

The following definition generalizes the order relation between intervals presented in the previous definition to interval-valued hesitant fuzzy sets.

Definition 18 *Let X be a non-empty set with cardinality N, $A, B \in IVHFS(X)$ such that:*

$$\mu_A(x) = \bigcup_{i=1}^{n_x^A} A_i^x = \bigcup_{i=1}^{n_x^A} [A_i^{x^L}, A_i^{x^U}] \text{ and } \mu_B(x) = \bigcup_{i=1}^{n_x^B} B_i^x = \bigcup_{i=1}^{n_x^B} [B_i^{x^L}, B_i^{x^U}]$$

for every $x \in X$, where for simplicity and without loss of generality it is supposed that the sets are ordered increasingly, i.e., $A_i^x \leq_{XY} A_{i+1}^x$, $B_i^x \leq_{XY} B_{i+1}^x$, and $A_i^x \cap A_j^x = \emptyset$ and $B_i^x \cap B_j^x = \emptyset$ for $i \neq j$.

Given $I \in FG_{n_I}([0,1])$, and S and H the following functions (score and ac-

curacy, respectively):

$$S(I) = \sum_{i=1}^{n_I} S(I_i) = \sum_{i=1}^{n_I} (I_i^U - I_i^L), \; H(I) = \frac{1}{n_I} \sum_{i=1}^{n_I} \left[\frac{I_i^L + I_i^U}{2} \right],$$

then, $A \le B$ if only if

(a) $H(\mu_A(x)) \le H(\mu_B(x))$ $\forall x \in X$ and $\exists x'$ s. t. $H(\mu_A(x')) < H(\mu_B(x'))$ or

(b) $H(\mu_A(x)) = H(\mu_B(x))$ $\forall x \in X$ and

(b1) $S(\mu_A(x)) \le S(\mu_B(x))$ $\forall x \in X$ and $\exists x'$ s. t. $S(\mu_A(x')) < S(\mu_B(x'))$ or

(b2) $S(\mu_A(x)) = S(\mu_B(x))$ $\forall x \in X$ and

(b2.1) $n_x^A \le n_x^B$ $\forall x \in X$ and $\exists x'$ s. t. $n_{x'}^A < n_{x'}^B$ or

(b2.2) $n_x^A = n_x^B, A_i^{x^U} \le B_i^{x^U}$ and $A_i^{x^L} \le B_i^{x^L}$, $\forall x \in X$ and $\forall i = 1, \ldots, n_x^A$.

Proposition 19 *Let $\le$ be the relation given in Definition 18. Then, $\le$ is an ordering relation for interval-valued hesitant fuzzy sets.*

Proof. In order to prove that this relation is an ordering relation, it must be proven that it is reflexive, antisymmetric and transitive.

(i) **Reflexivity:** it is obvious, as all the conditions in the definition of the relation are fulfilled with equalities (Condition $(b2.2)$).

(ii) **Antisymmetry:** given $A, B \in IVHFS(X)$ such that $A \le B$ and $B \le A$. Let us see that the only possible situation in $A = B$, distinguishing situations depending on the condition satisfied for each inequality.

- If $A \le B$ satisfies (a) and $B \le A$ satisfies (a):

$$H(\mu_A(x)) \le H(\mu_B(x)) \le H(\mu_A(x)), \; \forall x \in X,$$

  but it exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) \le H(\mu_A(x'))$, which is a contradiction.

- If $A \le B$ satisfies (a) and $B \le A$ satisfies (b),

$$H(\mu_A(x)) \le H(\mu_B(x)) = H(\mu_A(x)), \; \forall x \in X,$$

  but it exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) = H(\mu_A(x'))$, which is a contradiction.

- Analogously, it is proven that it is a contradiction for every combination unless $A \le B$ satisfies $(b2.2)$ and $B \le A$ satisfies $(b2.2)$, where

$$A_i^{x^L} \le B_i^{x^L} \le A_i^{x^L} \text{ and } A_i^{x^U} \le B_i^{x^U} \le A_i^{x^U},$$

  $\forall x \in X$ and $\forall i = 1, \ldots, n_x$. Therefore, $A_i^{x^L} = B_i^{x^L}$ and $A_i^{x^U} = B_i^{x^U}$, and then $A = B$.

(iii) **Transitivity:** given $A, B, C \in IVHFS(X)$ such that $A \le B$ and $B \le C$, let us see that $A \le C$.

- If $A \leq B$ satisfies (a) and $B \leq C$ satisfies (a):

$$H(\mu_A(x)) \leq H(\mu_B(x)) \leq H(\mu_C(x)),\ \forall x \in X,$$

but it exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) \leq H(\mu_C(x'))$, so A and C satisfies (a) and hence, $A \leq C$.

- If $A \leq B$ satisfies (a) and $B \leq C$ satisfies $(b1)$ or $(b2.1)$ or $(b2.2)$, then

$$H(\mu_A(x)) \leq H(\mu_B(x)) = H(\mu_C(x)),\ \forall x \in X,$$

but it exists $x' \in X$ such that $H(\mu_A(x')) < H(\mu_B(x')) = H(\mu_C(x'))$, so A and C satisfies (a) and hence, $A \leq C$.

- If $A \leq B$ satisfies $(b1)$ and $B \leq C$ satisfies $(b2.1)$ or $(b2.2)$: Both $A \leq B$ and $B \leq C$ satisfy (b), then

$$H(\mu_A(x)) = H(\mu_B(x)) = H(\mu_C(x)),\ \forall x \in X.$$

As $A \leq B$ satisfies (b1) and $B \leq C$ satisfies (b2.1), then

$$S(\mu_A(x)) \leq S(\mu_B(x)) = S(\mu_C(x))\ \forall x \in X.$$

In addition, $\exists x'$ s. t. $S(\mu_A(x')) < S(\mu_B(x'))$. As $S(\mu_B(x)) = S(\mu_C(x))\ \forall x \in X$, in particular $S(\mu_B(x')) = S(\mu_C(x'))$. Therefore,

$$\exists x' \text{ s. t. } S(\mu_A(x')) < S(\mu_B(x')) = S(\mu_C(x')).$$

Thus $A \leq C$ by (b1).

- If $A \leq B$ satisfies $(b2.1)$ and $B \leq C$ satisfies $(b2.2)$: Both $A \leq B$ and $B \leq C$ satisfy (b), then

$$H(\mu_A(x)) = H(\mu_B(x)) = H(\mu_C(x)),\ \forall x \in X.$$

As $A \leq B$ satisfies (b2.1) and $B \leq C$ satisfies (b2.2), then

$$S(\mu_A(x)) = S(\mu_B(x)) = S(\mu_C(x))\ \forall x \in X.$$

In addition,

$$n_x^A \leq n_x^B\ \forall x \in X \text{ and } \exists x' \text{ s. t. } n_{x'}^A < n_{x'}^B,$$

and

$$n_x^B = n_x^C, B_i^{x^U} \leq C_i^{x^U} \text{ and } B_i^{x^L} \leq C_i^{x^L} \forall x \in X \text{ and } \forall i = 1, \ldots, n_x^A.$$

Thus,

$$n_x^A \leq n_x^B = n_x^C\ \forall x \in X.$$

In addition, as $\exists x'$ s. t. $n_{x'}^A < n_{x'}^B$, and $n_x^B = n_x^C \forall x \in X$, then

$$n_{x'}^A < n_{x'}^B = n_{x'}^C.$$

That means $\exists x'$ s. t. $n_{x'}^{A} < n_{x'}^{C}$ and therefore $A \leq C$ because it satisfies (b2.1).

· If $A \leq B$ satisfies $(b2.2)$ and $B \leq C$ satisfies $(b2.2)$: Both $A \leq B$ and $B \leq C$ satisfy (b), then

$$H(\mu_A(x)) = H(\mu_B(x)) = H(\mu_C(x)),\ \forall x \in X.$$

As both $A \leq B$ and $B \leq C$ satisfy (b2.2) then

$$S(\mu_A(x)) = S(\mu_B(x)) = S(\mu_C(x))\ \forall x \in X.$$

In addition,

$$n_x^A = n_x^B, A_i^{x^U} \leq B_i^{x^U} \text{ and } A_i^{x^L} \leq B_i^{x^L} \forall x \in X \text{ and } \forall i = 1, \ldots, n_x^A$$

$$n_x^B = n_x^C, B_i^{x^U} \leq C_i^{x^U} \text{ and } B_i^{x^L} \leq C_i^{x^L} \forall x \in X \text{ and } \forall i = 1, \ldots, n_x^A.$$

Thus,

$$n_x^A = n_x^C, A_i^{x^U} \leq C_i^{x^U} \text{ and } A_i^{x^L} \leq C_i^{x^L} \forall x \in X \text{ and } \forall i = 1, \ldots, n_x^A$$

and therefore $A \leq C$, since (b2.2) is fulfilled. ■

One of the main interests of the interval-valued hesitant fuzzy sets lies on the fact that they generalize fuzzy sets and interval-valued fuzzy sets.

Remark 20 *The different types of sets previously given are related as follows:*

$$FS(X) \subseteq IVFS(X) \subseteq IVHFS(X) \subseteq T2FS(X),$$
$$FS(X) \subseteq THFS(X) \subseteq IVHFS(X) \subseteq T2FS(X).$$

Furthermore, interval-valued hesitant fuzzy sets are more manageable than 2-type fuzzy sets, due to the type of membership functions that define each one, which is a reason to work with the former instead of the latter.

In the next subsection, basic concepts about entropy with respect to different families of sets are treated, as well as the definition of dissimilarity measure, which is also analyzed in different situations.

2.3 Entropy and dissimilarity measure

The aim of an entropy is to quantify the uncertainty associated with either a fuzzy set or a generalization of it. In the next result, the definition of entropy for fuzzy sets is given. The definitions of entropy and dissimilarity measure in the classical fuzzy sets are well known and can be found in several sources, such as [13], given by Dubois and Prade.

Definition 21 *A mapping $E : FS(X) \to [0,1]$ is an entropy measure if it satisfies the following properties, where $A, B \in FS(X)$:*

(1) $E(A) = 0 \Leftrightarrow A$ *is crisp,*
(2) $E(A) = 1 \Leftrightarrow A$ *is the equilibrium set,*
(3) $E(A) = E(A^c)$*,*
(4) $E(A) \leq E(B)$ *if* $|\mu_A(x) - \mu_\xi(x)| \geq |\mu_B(x) - \mu_\xi(x)|,\ \forall x \in X$.

Dissimilarity measures are widely used in different fields. The usual definition in a fuzzy environment is given as follows.

Definition 22 *A mapping $D : FS(X) \times FS(X) \to [0,1]$ is a dissimilarity measure if it satisfies the following properties, where $A, B, C \in FS(X)$:*

(1) $D(A, B) = D(B, A)$*,*
(2) $D(A, A) = 0$*,*
(3) if $A \leq B \leq C$*, then* $D(A, B) \leq D(A, C)$ *and* $D(B, C) \leq D(A, C)$.

Some authors replace condition (2) by

(2) $D(A, B) = 0 \Leftrightarrow A = B$.

Some others consider a particular case of these dissimilarity measures, the ones obtained by considering the idea of restricted dissimilarity function given by Bustince et al. ([7,8]), that is, fulfilling the condition:

(4) $D(A, A^c) = 1$ iff A is a crisp set.

Thus, from now on, we will work only with restricted dissimilarities.

It must be noted that from a dissimilarity measure, a similarity measure is easily obtainable by $S(A, B) = h(D(A, B))$, with h monotone decreasing such that $h(1) = 0$ and $h(0) = 1$ (that is, for any negation). For this reason, it is enough to study just one of the two measures.

In [18], the authors adapt the entropy to Atanassov's intuitionistic fuzzy sets. This entropy is split into two functions, E_F and E_L, where each one represents a different meaning of entropy. The former describes the fuzziness of the set, i.e., it measures how similar it is to a crisp set. The latter function outlines the lack of knowledge, which shows the similarity with a fuzzy set.

Next definition provides the same concepts for interval-valued fuzzy sets based on tha fact that Atanassov's intuitionistic fuzzy sets and interval-valued fuzzy sets are mathematically equivalent (see [12]), so results given for one of them can be translated to the other.

Definition 23 *Let $E_F, E_L : IVFS(X) \to [0,1]$ be two mappings. The pair*

(E_F, E_L) is said to be a two-tuple entropy measure if E_F satisfies the following properties, where $A, B \in IVFS(X)$:

(1) $E_F(A) = 0 \Leftrightarrow A$ *is crisp or it is the pure interval-valued fuzzy set,*
(2) $E_F(A) = 1 \Leftrightarrow A$ *is the equilibrium set,*
(3) $E_F(A) = E_F(A^c)$,
(4) $E_F(A) \leq E_F(B)$ *if* $\forall x \in X$
$\mu_A(x) \leq_I \mu_B(x) \leq_I \mu_\xi(x)$ *for* $\mu_B^L(x) + \mu_B^U(x) \leq 1$ *or*
$\mu_\xi(x) \leq_I \mu_B(x) \leq_I \mu_A(x)$ *for* $\mu_B^L(x) + \mu_B^U(x) \geq 1$,

and E_L satisfies the following properties, where $A, B \in IVFS(X)$:

(1) $E_L(A) = 0 \Leftrightarrow A \in FS(X)$,
(2) $E_L(A) = 1 \Leftrightarrow A$ *is the pure interval-valued fuzzy set,*
(3) $E_L(A) = E_L(A^c)$,
(4) $E_L(A) \leq E_L(B)$ *if* $S(\mu_A(x)) \leq S(\mu_B(x))$, $\forall x \in X$, *where S denotes the score function given in Definition 17.*

This generalization to interval-valued fuzzy sets can be also carried out for the dissimilarity measure. This concept can be found in [17] for intuitionistic fuzzy sets, which is adapted to interval-valued fuzzy sets as follows.

Definition 24 *A mapping $D : IVFS(X) \times IVFS(X) \to [0, 1]$ is a (restricted) dissimilarity measure if it satisfies the following properties, where $A, B, C \in IVFS(X)$:*

(1) $D(A, B) = D(B, A)$,
(2) $D(A, A^c) = 1 \Leftrightarrow A$ *is crisp,*
(3) $D(A, B) = 0 \Leftrightarrow A = B$,
(4) if $A \leq B \leq C$, *then* $D(A, B) \leq D(A, C)$ *and* $D(B, C) \leq D(A, C)$.

It has been recently developed the definition of dissimilarity measure and entropy for interval-valued hesitant fuzzy sets, which can be found in [14].

Definition 25 *A mapping $D : IVHFS(X) \times IVHFS(X) \to [0, 1]$ is a (restricted) hesitant dissimilarity measure if it satisfies the following properties, where $A, B, C \in IVHFS(X)$:*

(1) $D(A, B) = D(B, A)$,
(2) $D(A, A^c) = 1 \Leftrightarrow A$ *is crisp,*
(3) $D(A, B) = 0 \Leftrightarrow A = B$,
(4) If $A \leq B \leq C$, *then* $D(A, B) \leq D(A, C)$ *and* $D(B, C) \leq D(A, C)$.

Example 26 *Some examples of hesitant dissimilarity measures were given by Xu and Xia ([30]). One of them was based on the Hamming dissimilarity and* *defined, for a finite set X with cardinality N, as follows:*

$$D_H(A,B) = \frac{1}{N}\sum_{x\in X}\left[\frac{1}{2n_x}\sum_{i=1}^{n_x}(|A_i^{x^L} - B_i^{x^L}| + |A_i^{x^U} - B_i^{x^U}|)\right],$$

for any $A, B \in IVHFS(X)$ where $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x = \bigcup_{i=1}^{n_x}[A_i^{x^L}, A_i^{x^U}]$ with $A_i^x \leq_{XY} A_{i+1}^x$ (with respect to the total ordering relation $\leq_{XY}$ associated to the score and accuracy functions given in Definition 17) for every $x \in X$ and $i \in \{1, \ldots, n_x - 1\}$, and analogously for the set B.

Definition 27 *Let $E : IVHFS(X) \to [0,1]$ be a mapping, and D a hesitant dissimilarity measure. E is said to be a hesitant entropy measure associated to D if it satisfies:*

(1) $E(A) = 0 \Leftrightarrow \mu_A(x) \subseteq \{0,1\},\ \forall x \in X$,
(2) $E(A) = 1 \Leftrightarrow A$ is the equilibrium set,
(3) $E(A) = E(A^c)$,
(4) $E(A) \leq E(B)$, if $D(A,\xi) \geq D(B,\xi)$.

In the next section, the proposal of this paper is developed, where a new definition of an entropy measure is given in an interval-valued hesitant fuzzy environment.

3 A new definition of entropy for interval-valued hesitant fuzzy sets

In the previous section, a definition of entropy for interval-valued hesitant fuzzy sets by Farhadinia (see [14]) was given. However, this definition only takes into account the distance to the equilibrium set, which may not be enough to quantify the uncertainty associated to an interval-valued hesitant fuzzy set.

In order to overcome this, a different definition of entropy is given. It is characterized by three mappings instead of just one, as Pal *et. al* (see [18]) did for Atanassov's intuitionistic fuzzy sets with two different mappings.

Hence, the developed entropy for interval-valued hesitant fuzzy sets is split into three functions: E_F, E_L and E_H. They are studied separately in the next three subsections, representing each one a different type of uncertainty associated to an interval-valued hesitant fuzzy set. This allows to provide a more detailed entropy measure.

3.1 Fuzziness entropy measure

The first function of the interval-valued hesitant fuzzy entropy represents the fuzziness of the set. The goal of this function is to measure how distant the set is from the union of a finite number of crisp sets. This mapping is similar to the one given by Definition 27, but with a modification in the first and last axioms, which are more efficient in order to build this part of the entropy.

Definition 28 *Let $E_F : IVHFS(X) \to [0,1]$ be a mapping. E_F is said to be a fuzziness entropy measure associated to a hesitant dissimilarity measure D if it satisfies the following properties, where $A, B \in IVHFS(X)$:*

(1) $E_F(A) = 0 \Leftrightarrow \mu_A(x) \in \{0, 1, \{0, 1\}, [0, 1]\},\ \forall x \in X$,
(2) $E_F(A) = 1 \Leftrightarrow A$ *is the equilibrium set,*
(3) $E_F(A) = E_F(A^c)$,
(4) $E_F(A) \le E_F(B)$, *if* $D(A_x, \xi) \ge D(B_x, \xi)\ \forall x \in X$, *where* $A_x = \{(y, \mu_A(x))|y \in X\}$ *and* $B_x = \{(y, \mu_B(x))|y \in X\}$.

The first axiom states that the fuzziness is null if the membership function is the union of crisp sets or the pure interval-valued fuzzy set. In the second axiom, the maximum fuzziness happens when the set is the equilibrium. The third one, requires that a set and its complement takes the same entropy. In the fourth axiom, two interval-valued hesitant fuzzy sets are compared with respect to E_F using the associated hesitant dissimilarity measure. In fact, the definition of fuzziness entropy is related to the dissimilarity, but it is not detailed in all the cases, since there is not ambiguity.

Furthermore, the local property can be given for this entropy measure in the case of finite referential sets. Firstly, some notation is necessary.

Definition 29 *Let X be a finite set with cardinality N. Given $M \subseteq \{1, \dots, N\}$ and $A = \{(x_i, \mu_A(x_i)|x_i \in X\} \in IVHFS(X)$, the interval valued hesitant fuzzy sets $A^{(M)}$ is defined as follows:*

$$A^{(M)} = \{(x_i, \mu_{A^{(M)}}(x_i))|x_i \in X\}$$

where

$$\mu_{A^{(M)}}(x_i) = \begin{cases} \mu_A(x_i) & \text{if } i \notin M, \\ \{0\} & \text{if } i \in M \text{ and } A_{x_i} \le \xi, \\ \{1\} & \text{if } i \in M \text{ and } A_{x_i} > \xi, \end{cases}$$

where $\le$ is the ordering relation given in Definition 18.

Remark 30 *It should be noted that as both sets A_{x_i} and ξ have constant membership functions (in $\mu_A(x_i)$ and $\{0.5\}$ respectively), $A_{x_i} \le \xi$ or $A_{x_i} > \xi$*

must hold.

To prove it, the only conflictive situation arises if $H(\mu_A(x_i)) = H(\{0.5\}) = 0.5$, $S(\mu_A(x_i)) = S(\{0.5\}) = 0$ *and* $n_{\mu_A(x_i)} = n_{\{0.5\}} = 1$. *From these conditions, the next equations must be satisfied:*

$$\mu_A(x_i)^U - \mu_A(x_i)^L = 0, \quad \mu_A(x_i)^U + \mu_A(x_i)^L = 1,$$

and the only possibility is that $\mu_A(x_i) = \{0.5\}$, *and as a consequence,* $A_{x_i} = \xi$.

It is clear that $A^{(M)}$ is only different to A in any $x_i \in X$ with $i \in M$. In the particular case $M = \{j\}$, the notation is simplified to $A^{(j)}$.

Definition 31 *Let* X *be a finite set with cardinality* N *and* $E_F : IVHFS(X) \to [0,1]$ *a fuzziness entropy measure.* E_F *is said to be a local fuzziness entropy measure if it exists a function* $f : FG([0,1]) \to [0,1]$ *such that for every* $x_j \in X$, *given* $A \in IVHFS(X)$:

$$E_F(A) - E_F(A^{(j)}) = f(\mu_A(x_j)),$$

or equivalently, it only depends on the term $\mu_A(x_j)$.

Remark 32 *It must be noted that* $E_F(A) - E_F(A^{(j)}) \in [0,1]$ *for all* $j = 1, \dots, n$. *To prove it, it is enough to see that* $D(A_{x_j}, \xi) \leq D(A^{(j)}_{x_j}, \xi)$, *as for the other* $x \in X$, *the equality is immediate.*

- *If* $A_{x_j} \leq \xi$, *then* $A^{(j)}_{x_j} = \{(x,0) | x \in X\} = \emptyset$. *Hence,* $\emptyset = A^{(j)}_{x_j} \leq A_{x_j} \leq \xi$. *By the last property in Definition 25,* $D(A_{x_j}, \xi) \leq D(\emptyset, \xi) = D(A^{(j)}_{x_j}, \xi)$, *and by the last condition of a fuzziness entropy,* $E_F(A^{(j)}) \leq E_F(A)$.
- *If* $A_{x_j} > \xi$, *then* $A^{(j)}_{x_j} = \{(x,1) | x \in X\} = X$. *Hence,* $\xi \leq A_{x_j} \leq A^{(j)}_{x_j} = X$. *By the last property in Definition 25,* $D(A_{x_j}, \xi) \leq D(X, \xi) = D(A^{(j)}_{x_j}, \xi)$, *and by the last condition of a fuzziness entropy,* $E_F(A^{(j)}) \leq E_F(A)$.

Henceforth, two results have been developed in order to ease the obtaining of local fuzziness entropy measures with functions whose properties are more manageable than the original ones in the definition of such entropy. Initially, we are going to characterize the local fuzziness entropies by means of the following result.

Theorem 33 *Let* X *be a finite set with cardinality* N, E_F *be the mapping* $E_F : IVHFS(X) \to [0,1]$ *and* D *a hesitant dissimilarity measure. Then,* E_F *is a local fuzziness entropy measure associated to* D *if and only if it exists a mapping* $h : FG([0,1]) \to [0,1]$ *such that*

$$E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)),$$

which also satisfies the following four axioms, given $I, J \in FG([0,1])$*:*

(1) $h(I) = 0 \Leftrightarrow I \in \{0, 1, \{0,1\}, [0,1]\}$,
(2) $h(I) = 1 \Leftrightarrow I = \mu_\xi(x)$,
(3) $h(I) = h(I^c)$,
(4) $h(I) \le h(J)$ *if* $D(X_I, \xi) \ge D(X_J, \xi)$, *where* $X_I = \{(x, I)|x \in X\}$ *and* $X_J = \{(x, J)|x \in X\}$.

Proof. First, let us suppose that E_F is a local fuzziness entropy, and by the definition of local for fuzziness entropy, it is known that it exists a function $f : FG([0,1]) \to [0,1]$ such that:

$$E_F(A) - E_F(A^{(j)}) = f(\mu_A(x_j)),\ \forall j \in \{1, \ldots, N\}.$$

Given $A \in IVHFS(X)$, applying the definition of local recursevely:

$$\begin{aligned} E_F(A) &= E_F(A^{(N)}) + f(\mu_A(x_N)) = \\ &= E_F((A^{(N)})^{(N-1)}) + f(\mu_{A^{(N)}}(x_{N-1}) + f(\mu_A(x_N)) = \\ &= E_F(A^{(N-1,N)}) + f(\mu_A(x_{N-1}) + f(\mu_A(x_N)) = \cdots = \\ &= E_F(A^{(1,\ldots,N)}) + \sum_{x \in X} f(\mu_A(x)). \end{aligned}$$

However, $A^{(1,\ldots,N)} = \{(x, \mu_{A^{(1,\ldots,N)}}(x)|x \in X, \mu_A(A^{(1,\ldots,N)})(x) \in \{0,1\}\}$, i.e., it is a crisp set, and therefore, by the first axiom of fuzziness entropy, $E_F(A^{(1,\ldots,N)}) = 0$. Hence,

$$E_F(A) = \sum_{x \in X} f(\mu_A(x)).$$

In addition, it is known that $E_F(A) \in [0,1]$ for every $A \in IVHFS(X)$. Then, for all $x_i \in X$, applying the mapping E_F to the set $X_{\mu_A(x_i)}$:

$$E_F(X_{\mu_A(x_i)}) = \sum_{x \in X} f(\mu_A(x_i)) = N f(\mu_A(x_i)) \in [0,1] \Rightarrow f(\mu_A(x_i)) \in [0, \frac{1}{N}].$$

Consequently, taking $h : FG([0,1]) \to [0,1]$ such that $h(I) = Nf(I)$, it is immediate that:

$$E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

Now, let us see that h satisfies the four conditions of the theorem.

(1) Given $I \in FG([0,1])$ and $X_I \in IVHFS(X)$, then:

$$E_F(X_I) = \frac{1}{N} \sum_{x \in X} h(I) = h(I),$$

so by the first axiom of Definition 28, it is known that:

$$E_F(X_I) = h(I) = 0 \quad \Leftrightarrow \quad \mu_A(x) = I \in \{0, 1, \{0, 1\}, [0, 1]\}.$$

(2) Given $I \in FG([0,1])$ and $X_I \in IVHFS(X)$ such that $E_F(X_I) = h(I)$. From the second axiom of E_F, it is obvious that:

$$E_F(X_I) = h(I) = 1 \quad \Leftrightarrow \quad \mu_A(x) = I = \{0.5\} = \mu_\xi(x).$$

(3) Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is obtained that $E_F(X_I) = h(I)$ and $E_F(X_{I^c}) = h(I^c)$, and as E_F satisfies the third axiom of Definition 28, $E_F(X_I) = E_F(X_{I^c})$ and hence,

$$h(I) = h(I^c).$$

(4) Given $I, J \in FG([0,1])$, and $X_I, X_J \in IVHFS(X)$, it is supposed that $D(X_I, \xi) \geq D(X_J, \xi)$, where by construction, $E_F(X_I) = h(I)$ and $E_F(X_J) = h(J)$. Due to E_F being a fuzziness entropy, the fourth axiom is satisfied and:

$$E_F(X_I) \leq E_F(X_J) \Leftrightarrow h(I) \leq h(J).$$

Now, in order to proceed with the second part of the proof, it is supposed that h satisfies the four conditions of the theorem, so it is necessary to prove that E_F is a local fuzziness entropy. First, let us see that it satisfies the four axioms of Definition 28:

(1) Given $A \in IVHFS(X)$:

$$0 = E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 0, \forall x \in X,$$

and as h satisfies the first property of the theorem, this only happens when:

$$\mu_A(x) \in \{0, 1, \{0, 1\}, [0, 1]\}, \ \forall x \in X.$$

(2) Given $A \in IVHFS(X)$:

$$1 = E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 1, \forall x \in X,$$

which is the same as $\mu_A(x) = 0.5, \forall x \in X$, as h fulfils the second axiom of the theorem.

(3) Given $A \in IVHFS(X)$ and A^c its complement, as h satisfies the third item of the theorem, it is known that $h(J) = h(J^c)$ for every finitely generated set, therefore:

$$E_F(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)^c) = \frac{1}{N} \sum_{x \in X} h(\mu_{A^c}(x)) = E_F(A^c).$$

(4) Let $A, B \in IVHFS(X)$ such that $D(A_x, \xi) \geq D(B_x, \xi)$ $\forall x \in X$. By the fourth axiom of the theorem for $I = \mu_A(x)$ and $J = \mu_B(x)$, $h(\mu_A(x)) \leq h(\mu_B(x))$ $\forall x \in X$, hence by construction of the mapping E_F:

$$E_F(A) \leq E_F(B).$$

In order to close the proof, let us see that it is also a local fuzziness entropy measure (Definition 31):

(L) Given $A \in IVHFS(X)$, for every $x_j \in X$:

$$\begin{aligned} &E_F(A) - E_F(A^{(j)}) = \\ &= \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) - \frac{1}{N}\left(\sum_{x\in X\setminus\{x_j\}} h(\mu_A(x)) + h(\mu_{A^{(j)}}(x_j))\right) = \\ &= \frac{1}{N}(h(\mu_A(x_j)) - h(\mu_{A^{(j)}}(x_j)) = \frac{1}{N}h(\mu_A(x_j)) = f(\mu_A(x_j)), \end{aligned}$$

i.e., it only depends on the term $\mu_A(x_j)$ for every j as $\mu_{A^{(j)}}(x_j) \in \{0,1\}$ and by hypothesis, $h(\mu_{A^{(j)}}(x_j)) = 0$. Therefore, it is local. ■

After the simplification provided by the previous theorem, the next result allows to go another step forward and ease even more the obtaining of a local fuzziness entropy.

Corollary 34 *Let X be a finite set with cardinality N, E_F be the mapping $E_F : IVHFS(X) \to [0,1]$ and D a hesitant dissimilarity measure where $D(A,\xi)$ is defined in function of the terms $|A_i^{x^U} - 0.5|$ and $|A_i^{x^L} - 0.5|$, and where $D(A,\xi) = 0.5$ if and only if $A_i^{x^L}, A_i^{x^U} \in \{0,1\}$ for every $x \in X$ and $i \in \{1,\ldots,n_x\}$.*

Then, E_F is a local fuzziness entropy associated to D if and only if it exists a mapping $g : [0,1] \to [0,1]$ such that

$$E_F(A) = \frac{1}{N}\sum_{x\in X} g(2D(A_x,\xi)),$$

which also satisfies the following properties:

(1) $g(a) = 0 \Leftrightarrow a = 1$,
(2) $g(a) = 1 \Leftrightarrow a = 0$,
(3) g is monotone decreasing.

Proof. It is enough to see that the function $h(I) = g(2D(X_I,\xi))$ satisfies the four axioms in Theorem 33, and the result will be proven.

(1) Let $I \in FG([0,1])$ such that $h(I) = g(2D(X_I,\xi)) = 0$, and by the second

axiom that g satisfies:

$$h(I) = g(2D(X_I, \xi)) = 0 \Leftrightarrow 2D(X_I, \xi) = 1 \Leftrightarrow D(X_I, \xi) = 0.5.$$

Given $I = I_1 \cup \cdots \cup I_{n_I} \in FG_{n_I}([0,1])$ with $I_i = [I_i^L, I_i^U]$ $\forall i$, by the hypothesis about D, this only happens when $I_i^L, I_i^U \in \{0,1\}$ $\forall i$, or what is the same, $I_i \in \{0, 1, [0,1]\}$ $\forall i$. Equivalently, $I \in \{0, 1, \{0,1\}, [0,1]\}$.

(2) Given $I \in FG([0,1])$ such that $h(I) = g(2D(X_I, \xi)) = 1$. This only holds when $D(X_I, \xi) = 0$ for the first axiom that g satisfies, and by the definition of hesitant dissimilarity (third axiom of Definition 25) $I = \{0, 5\} = \mu_\xi(x)$.

(3) Given $I \in FG([0,1])$, as 0.5 is the center of the interval $[0,1]$, by symmetry and the hypothesis about how D is defined, $D(X_I, \xi) = D(X_{I^c}, \xi)$, and it is immediate that $g(2D(X_I, \xi)) = g(2D(X_{I^c}, \xi))$.

(4) Given $I, J \in FG([0,1])$ such that $D(X_I, \xi) \geq D(X_J, \xi)$, as g is monotone decreasing by the third axiom:

$$h(I) = g(2D(X_I, \xi)) \leq g(2D(X_J, \xi)) = h(J). \quad \blacksquare$$

These two last results allow to obtain fuzziness entropies given by Definition 28 in an simpler way, where it is only needed a hesitant dissimilarity and a function h satisfying the three conditions from Corollary 34, which are much more manageable than the original ones.

In order to illustrate this first part of the entropy, an example is given next, where a particular dissimilarity and function g are selected as in Corollary 34.

Example 35 *Let X be a finite set with cardinality N, and $E_F : IVHFS(X) \to [0,1]$ given by:*

$$E_F(A) = \frac{1}{N} \sum_{x \in X} \left[1 - 2D_H(\mu_A(x), \{0.5\})\right],$$

and where D_H is the hesitant normalized Hamming dissimilarity, which was first developed by [30] for hesitant fuzzy sets, and adapted to interval-valued hesitant fuzzy sets by [14]. The dissimilarity for finite interval-valued hesitant fuzzy sets has the expression given in Example 26.

Then, E_F is a local fuzziness entropy measure, as it is a particular situation of the Corollary 34, where $g(a) = 1 - a$ and $D = D_H$, both satisfying the required properties.

3.2 Lack of knowledge entropy measure

The second part of the entropy definition is given by a function which represents the lack of knowledge. The distance of the set to the union of a finite number of classical fuzzy sets is measured by this function. Thus, a different kind of uncertainty is considered in this case.

Using the same notation as in the previous subsection, this function is defined as follows:

Definition 36 *Let $E_L : IVHFS(X) \to [0,1]$ be a mapping. E_L is said to be a lack of knowledge entropy measure if it satisfies the following properties, where $A, B \in IVHFS(X)$ with $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x = \bigcup_{i=1}^{n_x} [A_i^{x^L}, A_i^{x^U}] \in FG_{n_x}([0,1])$ $\forall x \in X$, and respectively for B:*

(1) $E_L(A) = 0 \Leftrightarrow S(A_i^x) = 0$, $\forall i = 1, \ldots, n_x$, $\forall x \in X$,
(2) $E_L(A) = 1 \Leftrightarrow A$ is the pure interval-valued fuzzy set,
(3) $E_L(A) = E_L(A^c)$,
(4) $E_L(A) \leq E_L(B)$ if $\forall x \in X$ $S(\mu_A(x)) \leq S(\mu_B(x))$, where S denotes the score function given in Definition 18.

The first axiom states what has been already mentioned in the first part of this subsection, a null entropy is given when all the sets A_i^x are singletons, i.e., the set A is a classical fuzzy set. The maximum entropy is found when A is the pure interval-valued fuzzy set. In the third point, the entropy of a set and its complement must match. In the last axiom, it is given how to compare two interval-valued hesitant fuzzy sets with respect to the lack of knowledge entropy measure, where it is taken into account the upper ($A_i^{x^U}$) and lower ($A_i^{x^L}$) bounds of each A_i^x for every $i = 1, \ldots, n_x$ and $x \in X$.

As it has been done for the fuzziness entropy in the previous subsection, the concept of local lack of knowledge is also studied.

Definition 37 *Let X be a finite set with cardinality N and $E_L : IVHFS(X) \to [0,1]$ a lack of knowledge entropy measure. E_L is said to be a local lack of knowledge entropy measure if it exists a function $f : FG([0,1]) \to [0,1]$ such that for every $x_j \in X$, given $A \in IVHFS(X)$:*

$$E_L(A) - E_L(A^{(j)}) = f(\mu_A(x_j)),$$

or equivalently, it only depends on the term $\mu_A(x_j)$.

Remark 38 *It must be noted that $E_L(A) - E_L(A^{(j)}) \in [0,1]$ for all $j = 1, \ldots, n$. By construction, $S(\mu_A(x)) = S(\mu_{A^{(j)}}(x))$, $\forall x \neq x_j$. Furthermore,*

$S(\mu_{A^{(j)}}(x_j)) = 0$, so it is obvious that $S(\mu_A(x_j)) \geq S(\mu_{A^{(j)}}(x_j))$, and by the last axiom of a lack of knowledge entropy, $E_L(A) \geq E_L(A^{(j)})$.

From here on out, two results are given in order to obtain local lack of knowledge entropy measures with lighter conditions, with functions whose properties are more manageable than the ones of the original definition.

Theorem 39 *Let X be a finite set with cardinality N and E_L be the mapping $E_L : IVHFS(X) \to [0,1]$. Then, E_L is a local lack of knowledge entropy measure if and only if it exists a mapping $h : FG([0,1]) \to [0,1]$ such that*

$$E_L(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

which also satisfies the following four axioms, given $I, J \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_{n_I} \in FG_{n_I}([0,1])$ and $J = J_1 \cup \cdots \cup J_{n_J} \in FG_{n_J}([0,1])$ with $I_i = [I_i^L, I_i^U]$ $\forall i$ and respectively for J:

(1) $h(I) = 0 \Leftrightarrow S(I_i) = 0, \ \forall i = 1, \ldots, n_I$,
(2) $h(I) = 1 \Leftrightarrow I = [0,1]$,
(3) $h(I) = h(I^c)$,
(4) $h(I) \leq h(J)$ if $S(I) \leq S(J)$.

Proof. First, let us suppose that E_L is a local lack of knowledge entropy, and by the definition of local for lack of knowledge entropy, it is known that it exists a function $f : FG([0,1]) \to [0,1]$ such that:

$$E_L(A) - E_L(A^{(j)}) = f(\mu_A(x_j)), \ \forall j \in \{1, \ldots, N\}.$$

Given $A \in IVHFS(X)$, applying the definition of local recursevely:

$$E_L(A) = \cdots = E_L(A^{(1,\ldots,N)}) + \sum_{x \in X} f(\mu_A(x)).$$

However, $A^{(1,\ldots,N)} = \{(x, \mu_{A^{(1,\ldots,N)}}(x) | x \in X, \mu_A(A^{(1,\ldots,N)})(x) \in \{0,1\}\}$, i.e., the score function applied on any of them is equal to 0, and by the first axiom of lack of knowledge entropy, $E_L(A^{(1,\ldots,N)}) = 0$. Hence,

$$E_L(A) = \sum_{x \in X} f(\mu_A(x)).$$

In addition, it is known that $E_L(A) \in [0,1]$ for every $A \in IVHFS(X)$. Then, for all $x_i \in X$, applying the mapping E_L to the set $X_{\mu_A(x_i)}$:

$$E_L(X_{\mu_A(x_i)}) = \sum_{x \in X} f(\mu_A(x_i)) = N f(\mu_A(x_i)) \in [0,1] \Rightarrow f(\mu_A(x_i)) \in [0, \frac{1}{N}].$$

Consequently, taking $h : FG([0,1]) \to [0,1]$ such that $h(I) = Nf(I)$, it is immediate that:

$$E_L(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

Now, let us see that h satisfies the four conditions of the theorem.

(1) Given $I \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_{n_I}$, let us take $X_I \in IVHFS(X)$ such that $\mu_{X_I}(x) = I$ for all $x \in X$. Then:

$$E_L(X_I) = \frac{1}{N} \sum_{x \in X} h(\mu_{X_I}(x)) = \frac{1}{N} \sum_{i=1}^{N} h(I) = h(I),$$

and therefore $h(I) = 0 \Leftrightarrow E_L(X_I) = 0$. E_L satisfies the first axiom of lack of knowledge entropy, so $h(I) = 0 \Leftrightarrow S(I_i) = 0$, for all $i = 1, \ldots, n_I$ and the first axiom is proved.

(2) Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is direct that

$$h(I) = 1 \Leftrightarrow E_L(X_I) = 1,$$

and as E_L satisfies the second axiom of Definition 36, $I = [0,1]$.

(3) Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is obtained that $E_L(X_I) = h(I)$ and $E_L(X_{I^c}) = h(I^c)$, and as E_L satisties the third axiom of a lack of knowledge entropy, $E_L(X_I) = E_L(X_I^c) = E_L(X_{I^c})$ and hence,

$$h(I) = h(I^c).$$

(4) Let $I, J \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_{n_I}$ and $J = J_1 \cup \cdots \cup J_{n_J}$, and $S(I) \leq S(J)$.

Given $X_I, X_J \in IVHFS(X)$, as $S(I) \leq S(J)$ and E_L satisfies the fourth axiom of the lack of knowledge entropy, $E_L(X_I) \leq E_L(X_J)$. However, $E_L(X_I) = h(I)$ and $E_L(X_J) = h(J)$, so

$$h(I) \leq h(J).$$

Now, in order to proceed with the second part of the proof, it is supposed that h satisfies the four conditions of the theorem, so it is needed to prove that E_L is a local lack of knowledge entropy. First, let us proof the four conditions of Definition 36:

(1) Given $A \in IVHFS(X)$,

$$0 = E_L(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 0, \ \forall x \in X,$$

and as h satisfies (1), then $S(\mu_A(x)) = 0,\ \forall x \in X$, and hence, it is a finite union of singletons.

(2) Given $A \in IVHFS(X)$,

$$1 = E_L(A) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) \Leftrightarrow \sum_{x\in X} h(\mu_A(x)) = N,$$

and it is known by definition that $h(I) \in [0,1]$ for every finitely generated set, so the only possible situation is that

$$h(\mu_A(x)) = 1 \Leftrightarrow \mu_A(x) = [0,1],\ \forall x \in X.$$

(3) Given $A \in IVHFS(X)$ and A^c its complement, as h satisfies the third property of the theorem, it is known that $h(J) = h(J^c)$ for every finitely generated set, therefore:

$$E_L(A) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)^c) = \frac{1}{N}\sum_{x\in X} h(\mu_{A^c}(x)) = E_L(A^c).$$

(4) Given $A, B \in IVHFS(X)$ such that $\forall x \in X$:

$$S(\mu_A(x)) \leq S(\mu_B(x)).$$

From the last inequality, as h satisfies the fourth axiom of the theorem, $h(\mu_A(x)) \leq h(\mu_B(x))$. Therefore:

$$E_L(A) = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) \leq \frac{1}{N}\sum_{x\in X} h(\mu_B(x)) = E_L(B).$$

Finally, it must be proven that E_L is also local:

(L) Given $A \in IVHFS(X)$, for every $x_j \in X$:

$$\begin{aligned} & E_L(A) - E_L(A^{(j)}) = \\ & = \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) - \frac{1}{N}\left(\sum_{x\in X\setminus\{x_j\}} h(\mu_A(x)) + h(\mu_{A^{(j)}}(x_j))\right) = \\ & = \frac{1}{N}(h(\mu_A(x_j)) - h(\mu_{A^{(j)}}(x_j)) = \frac{1}{N}h(\mu_A(x_j)) = f(\mu_A(x_j)), \end{aligned}$$

i.e., it only depends on the term $\mu_A(x_j)$ for every j as $\mu_{A^{(j)}}(x_j) \in \{0,1\}$ and by hypothesis, $h(\mu_{A^{(j)}}(x_j)) = 0$. Therefore, it is local. ■

With the support of the previous result, the next corollary allows to get local lack of knowledge entropies by a mapping with simpler achievable conditions.

Corollary 40 *Let X be a finite set with cardinality N and E_L be the mapping $E_L : IVHFS(X) \to [0,1]$. Then, E_L is a local lack of knowledge entropy if and only if it exists a mapping $g : [0,1] \to [0,1]$ such that*

$$E_L(A) = \frac{1}{N} \sum_{x \in X} g(S(\mu_A(x))),$$

which also satisfies the following properties:

(1) $g(a) = 0 \Leftrightarrow a = 0$,
(2) $g(a) = 1 \Leftrightarrow a = 1$,
(3) g *is monotone increasing.*

Proof. It is enough to see that the function $h(I) = g(S(I))$ fulfils the four conditions of Theorem 39, and the result would be proven.

(1) Given $I \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_{n_I}$:

$$h(I) = 0 = g(S(I)),$$

but for the first property that g satisfies, $g(a) = 0 \Leftrightarrow a = 0$, and then $S(I) = 0$.

(2) Given $I \in FG([0,1])$

$$h(I) = 1 = g(S(I)),$$

and for the second property of g, $S(I) = 1$, which only happens when $I = [0,1]$.

(3) Given $I \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_n$, and the complement $I^c = I_1^c \cup \cdots \cup I_n^c$. Given any i:

$$I_i^c = [1 - I_i^U, 1 - I_i^L],$$

so

$$S(I_i^c) = I_i^U - I_i^L = S(I_i).$$

Bearing this in mind, $h(I) = h(I^c)$.

(4) Given $I, J \in FG([0,1])$ such that $I = I_1 \cup \cdots \cup I_{n_I}$ and $J = J_1 \cup \cdots \cup J_{n_J}$, with $S(I) \leq S(J)$. The third property states that g is monotone increasing, hence:

$$g(S(I)) \leq g(S(J)) \Leftrightarrow h(I) \leq h(J). \quad \blacksquare$$

With this two last results, it has been found a way to obtain local lack of knowledge entropy measures just with a mapping g satisfying the properties of Theorem 40, which are less complicated to obtain than the ones in the original definition of this entropy measure.

As it has been done with the first mapping, an example is given next, starting from the last corollary.

Example 41 *Let X be a finite set with cardinality N, and $E_L : IVHFS(X) \to [0,1]$ given by:*

$$E_L(A) = \frac{1}{N} \sum_{x \in X} \sum_{i=1}^{n_x} S(A_i^x),$$

where $\mu_A(x) = A_1^x \cup \cdots \cup A_{n_x}^x \in FG_{n_x}([0,1])$, $\forall x \in X$, with $A_i^x = [A_i^{x^L}, A_i^{x^U}]$, $\forall i$.

This is obviously a local lack of knowledge entropy, as it is the particular case of Corollary 40 with $g(a) = a$.

3.3 Hesitance entropy measure

The last part of the definition of entropy in an interval-valued hesitant environment is given by a function which measures the distance of a set to a single interval-valued fuzzy set. It has been called hesitance, and it is defined as follows:

Definition 42 *Let $E_H : IVHFS(X) \to [0,1]$ be a mapping. E_H is said to be a hesitance entropy measure if it satisfies the following properties, where $A, B \in IVHFS(X)$:*

(1) $E_H(A) = 0 \Leftrightarrow A \in IVFS(X)$,
(2) $\lim_{n_x^A \to \infty} E_H(A) = 1 \ \forall x \in X$,
(3) $E_H(A) = E_H(A^c)$,
(4) $E_H(A) \leq E_H(B)$ *if* $\forall x \in X$:

$$n_x^A \leq n_x^B,$$

where

$$\mu_A(x) = \bigcup_{i=1}^{n_x^A} A_i^x \quad \text{and} \quad \mu_B(x) = \bigcup_{i=1}^{n_x^B} B_i^x \ \forall x \in X,$$

i.e., n_x^A and n_x^B represent the number of disjoint intervals that shapes the set $\mu_A(x)$ and $\mu_B(x)$ respectively.

As it has been already said, a null entropy happens when the set is an interval-valued fuzzy one. The second axiom remarks that the entropy tends to its maximum when the number of sets defining $\mu_A(x)$ for each point tends to infinite. In this axiom there is an abuse of notation: since A is fixed, also n_x^A is; but with this expression we would like to say that, for any x, if we consider the infimum of the values of the entropies of the sets with n disjoint intervalar components, the limit when n tends to infinity is equal to 1. The third one, states that a set and its complement must have the same entropy value. In the latter property, a set is greater than another with respect to this entropy when for every point, the number of intervals defining the set is also greater.

Next, an extension of this definition is given, adding the property of local to hesitance entropy measures.

Definition 43 *Let X be a finite set with cardinality N and $E_H : IVHFS(X) \to [0,1]$ a hesitance entropy measure. E_H is said to be a local hesitance entropy measure if it exists a function $f : FG([0,1]) \to [0,1]$ such that for every $x_j \in X$, given $A \in IVHFS(X)$:*

$$E_H(A) - E_H(A^{(j)}) = f(\mu_A(x_j)),$$

or equivalently, it only depends on the term $\mu_A(x_j)$.

Remark 44 *It must be noted that $E_H(A) - E_H(A^{(j)}) \in [0,1]$ for all $j = 1, \ldots, n$. By construction, $n_x^A = n_x^{A^{(j)}}$, $\forall x \neq x_j$. Furthermore, $n_{x_j}^{A^{(j)}} = 1$, so it is obvious that $n_{x_j}^A \geq n_{x_j}^{A^{(j)}}$, and by the last axiom of a hesitance entropy, $E_H(A) \geq E_H(A^{(j)})$.*

The next two results that are about to be developed, provide a way to obtain local hesitance entropies with simpler conditions, avoiding the more complex ones in the original definition previously given.

Theorem 45 *Let X be a finite set with cardinality N and E_H be the mapping $E_H : IVHFS(X) \to [0,1]$. Then, E_H is a local hesitance entropy measure if and only if it exists a mapping $h : FG([0,1]) \to [0,1]$ such that*

$$E_H(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

which also satisfies the following four axioms, given $I, J \in FG([0,1])$ such that $I \in FG_{n_I}([0,1])$ and $J \in FG_{n_J}([0,1])$:

(1) $h(I) = 0 \Leftrightarrow n_I = 1$,
(2) $\lim_{n_I \to \infty} h(I) = 1$,
(3) $h(I) = h(I^c)$,
(4) $h(I) \leq h(J)$ *if* $n_I \leq n_J$.

Proof. First, let us suppose that E_H is a local hesitance entropy, and by the definition of local for hesitance entropy, it is known that it exists a function $f : FG([0,1]) \to [0,1]$ such that:

$$E_H(A) - E_H(A^{(j)}) = f(\mu_A(x_j)), \ \forall j \in \{1, \ldots, N\}.$$

Given $A \in IVHFS(X)$, applying the definition of local recursevely:

$$E_H(A) = \cdots = E_H(A^{(1,\dots,N)}) + \sum_{x \in X} f(\mu_A(x)).$$

However, $A^{(1,\dots,N)} = \{(x, \mu_{A^{(1,\dots,N)}}(x) | x \in X, \mu_A(A^{(1,\dots,N)})(x) \in \{0,1\}\}$, i.e., $n_x^A = 1 \; \forall x \in X$, and by the first axiom of hesitance entropy, $E_H(A^{(1,\dots,N)}) = 0$. Hence,

$$E_H(A) = \sum_{x \in X} f(\mu_A(x)).$$

In addition, it is known that $E_H(A) \in [0,1]$ for every $A \in IVHFS(X)$. Then, for all $x_i \in X$, applying the mapping E_H to the set $X_{\mu_A(x_i)}$:

$$E_H(X_{\mu_A(x_i)}) = \sum_{x \in X} f(\mu_A(x_i)) = Nf(\mu_A(x_i)) \in [0,1] \Rightarrow f(\mu_A(x_i)) \in [0, \frac{1}{N}].$$

Consequently, taking $h : FG([0,1]) \to [0,1]$ such that $h(I) = Nf(I)$, it is immediate that:

$$E_H(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)).$$

Now, let us see that h satisfies the four conditions of the theorem.

(1) Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$ and $X_I \in IVHFS(X)$ such that $\mu_{X_I}(x) = I$ for all $x \in X$. Then:

$$E_H(X_I) = \frac{1}{N} \sum_{x \in X} h(\mu_{X_I}(x)) = \frac{1}{N} \sum_{x \in X} h(I) = h(I),$$

and therefore $h(I) = 0 \Leftrightarrow E_H(X_I) = 0$. E_H satisfies the first axiom of a hesitance entropy, so $h(I) = 0 \Leftrightarrow n_I = 1$, and the first axiom is proved.

(2) Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$, and $X_I \in IVHFS(X)$, it is direct that

$$h(I) = 1 \Leftrightarrow E_H(X_I) = 1,$$

and as E_H satisfies the second axiom of a hesitance entropy measure, $\lim_{n_x \to \infty} E_H(X_I) = 1, \; x \in X$, and by definition, $n_I = n_x$, so $\lim_{n_I \to \infty} h(I) = 1$.

(3) Given $I \in FG([0,1])$, and $X_I \in IVHFS(X)$, it is obtained that $E_H(X_I) = h(I)$ and $E_H(X_{I^c}) = h(I^c)$, and as E_H satistics the third axiom of a hesitance entropy, $E_H(X_I) = E_H(X_I^c) = E_H(X_{I^c})$ and hence,

$$h(I) = h(I^c).$$

(4) Let $I, J \in FG([0,1])$ such that $I \in FG_{n_I}([0,1])$ and $J \in FG_{n_J}([0,1])$ and $n_I \le n_J$, and $X_I, X_J \in IVHFS(X)$. By the last axiom of a hesitance entropy and the hypothesis $n_I \le n_J$, $E_H(X_I) \le E_H(X_J)$, and by definition of both interval-valued hesitant fuzzy sets, $h(I) \le h(J)$.

Now, in order to proceed with the second part of the proof, it is supposed that h satisfies the four conditions of the theorem, so it is needed to prove that E_H is a local hesitance entropy. On one hand, the properties of Definition 42 must be proven:

(1) Given $A \in IVHFS(X)$,

$$0 = E_H(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \Leftrightarrow h(\mu_A(x)) = 0, \ \forall x \in X,$$

and as h satisfies (1), then $n_x = 1$, $\forall x \in X$ where $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x$, or equivalently, $A \in IVFS(X)$.

(2) Given $A \in IVHFS(X)$, and by the second condition of the theorem, $\lim_{n_I \to \infty} h(I) = 1$, for every finitely generated set. Therefore:

$$\forall x \in X \lim_{n_x \to \infty} E_H(A) = \lim_{n_x \to \infty} \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) = \frac{1}{N} \sum_{x \in X} \lim_{n_x \to \infty} h(\mu_A(x)) = 1.$$

(3) Given $A \in IVHFS(X)$ and A^c its complement. As h satisfies the third property of the theorem, it is known that $h(J) = h(J^c)$ for every finitely generated set, therefore:

$$E_H(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)^c) = \frac{1}{N} \sum_{x \in X} h(\mu_{A^c}(x)) = E_H(A^c).$$

(4) Given $A, B \in IVHFS(X)$ where $\mu_A(x) = \bigcup_{i=1}^{n_x^A} A_i^x$ and $\mu_B(x) = \bigcup_{i=1}^{n_x^B} B_i^x$, $\forall x \in X$. Let us suppose that $n_x^A \leq n_x^B$ $\forall x \in X$, and by the fourth axiom that h satisfies:

$$h(\mu_A(x)) \leq h(\mu_B(x)), \ \forall x \in X,$$

and by construction of the mapping E_H:

$$E_H(A) = \frac{1}{N} \sum_{x \in X} h(\mu_A(x)) \leq \frac{1}{N} \sum_{x \in X} h(\mu_B(x)) = E_H(B).$$

On the other hand, let us prove that E_H is also local:

(L) Given $A \in IVHFS(X)$, for every $x_j \in X$:

$$E_H(A) - E_H(A^{(j)}) =$$
$$= \frac{1}{N}\sum_{x\in X} h(\mu_A(x)) - \frac{1}{N}\left(\sum_{x\in X\setminus\{x_j\}} h(\mu_A(x)) + h(\mu_{A^{(j)}}(x_j))\right) =$$
$$= \frac{1}{N}(h(\mu_A(x_j)) - h(\mu_{A^{(j)}}(x_j)) = \frac{1}{N}h(\mu_A(x_j)) = f(\mu_A(x_j)),$$

i.e., it only depends on the term $\mu_A(x_j)$ for every j as $\mu_{A^{(j)}}(x_j) \in \{0,1\}$ and by hypothesis, $h(\mu_{A^{(j)}}(x_j)) = 0$. Therefore, it is local. ■

The next result provides another step forward to simplify the conditions required to obtain a local hesitance entropy measure, where a new mapping is used to get it. Before the corollary, some notation is needed.

Remark 46 *The mapping $NInt : FG([0,1]) \to \mathbb{N}$ provides the number of closed disjoint subintervals that shape the finitely generated set. Given $I = \bigcup_{i=1}^{n_I} I_i \in FG([0,1])$, $NInt(I) = n_I$.*

Corollary 47 *Let X be a finite set with cardinality N and E_H be the mapping $E_H : IVHFS(X) \to [0,1]$. Then, E_H is a local hesitance entropy if and only if it exists a mapping $g : \mathbb{N} \to [0,1]$ such that*

$$E_H(A) = \frac{1}{N}\sum_{x\in X} g(NInt(\mu_A(x))),$$

which also satisfies the following properties:

(1) $g(a) = 0 \Leftrightarrow a = 1$,
(2) $\lim_{a\to\infty} g(a) = 1$,
(3) g is monotone increasing.

Proof. To prove the result, it is enough to see that the function $h(I) = g(NInt(I))$ satisfies the four axioms in Theorem 45.

(1) Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$:

$$h(I) = g(NInt(I)) = 0,$$

but for the first property that g satisfies, $g(a) = 0 \Leftrightarrow a = 1$, and then $NInt(I) = n_I = 1$.

(2) Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$:

$$\lim_{n_I\to\infty} h(I) = \lim_{n_I\to\infty} g(NInt(I)) = \lim_{n_I\to\infty} g(n_I),$$

but for the second property of g, $\lim_{a\to\infty} g(a) = 1$, and then $\lim_{n_I\to\infty} h(I) = 1$.

(3) Given $I, I^c \in FG_{n_I}([0,1]) \subseteq FG([0,1])$ a finitely generated set I and its complement I^c, as both are generated by the same number of closed disjoint intervals:

$$h(I) = g(NInt(I)) = g(NInt(I^c)) = h(I^c).$$

(4) Given $I \in FG_{n_I}([0,1]) \subseteq FG([0,1])$ and $J \in FG_{n_J}([0,1]) \subseteq FG([0,1])$ such that $n_I \leq n_J$, it is known by the increasing monotony of g that:

$$h(I) = g(NInt(I)) = g(n_I) \leq g(n_J) = g(NInt(J)) = h(J). \quad \blacksquare$$

As it has been done with the previous two mappings of this new definition of hesitant entropy, these last two results allow to get rid of the difficulties associated to the third part of the entropy with functions which are easier to obtain than the one in the original definition.

Again, a brief example is shown next to illustrate an obtainable particular case of local hesitance entropy by these last two results.

Example 48 *Let X be a finite set with cardinality N. Let $E_H : IVHFS(X) \to [0,1]$ be given by:*

$$E_H(A) = \frac{1}{N} \sum_{x \in X} (1 - \frac{1}{n_x}),$$

where $\mu_A(x) = \bigcup_{i=1}^{n_x} A_i^x, \ \forall x \in X$.

Then, E_H is a local hesitance entropy, as it is the particular case of Corollary 47 with $g(a) = 1 - \frac{1}{a}$.

Once that the three mappings have been defined and studied separately in the last three subsections, the joint definition of entropy is given and analized in the next one.

3.4 Joint hesitant entropy measure

The definition of the hesitant entropy proposed in this work is given next, where the three mappings E_F, E_L and E_H are put together in order to measure different types of uncertainties associated to a hesitant fuzzy set.

Definition 49 *Let $E_F, E_L, E_H : IVHFS(X) \to [0,1]$ be three mappings. The triplet (E_F, E_L, E_H) is said to be a* **joint entropy measure in an interval-valued hesitant fuzzy environment** *if E_F, E_L and E_H satisfy the axioms of Definitions 28, 36 and 42 and the local properties of Definitions 31, 37 and 43, respectively.*

The usual situation where interval-valued hesitant fuzzy sets can be applied arises when the evaluation of certain alternatives $(A_1, \ldots, A_n)$ with respect to some parameters $(x_1, \ldots, x_N)$ are given by several experts, whose opinions are summed up in a single interval-valued hesitant fuzzy set for each alternative and parameter.

In order to illustrate this and the way that entropy works and varies depending on the type of interval-valued hesitant fuzzy sets used, the next example has been carried out.

Example 50 *Let us supose that a business needs to hire a building company to carry out a construction. The business receives four different proposals* (A_1, A_2, A_3, A_4)*. Each one is evaluated by four different parameters* (x_1, x_2, x_3, x_4) *by three experts, whose opinion are summed up in interval-valued hesitant fuzzy sets. Then, the set that we are working with has four elements, i.e.,* $X = \{x_1, x_2, x_3, x_4\}$.

Let us obtain a joint entropy measure (E_F, E_L, E_H) *through the different results developed in the previous sections, specificly, Corollaries 34, 40 and 47 for* E_F, E_L *and* E_H *respectively. Given* $A \in IVHFS(X)$ *defined by* $\mu_A(x_i) = \bigcup_{j=1}^{n_{x_i}} A_j^{x_i} = \bigcup_{j=1}^{n_{x_i}} [A_j^{x_i^L}, A_j^{x_i^U}]$, $\forall x_i \in X$*:*

- *For* E_F*, the dissimilarity measure selected is the hesitant normalized Hamming dissimilarity, defined previously in Example 26, as well as the function* $g(a) = 1 - a^2$*, which satisfies the properties of Corollary 34. The local fuzziness entropy obtained is given as:*

$$E_F(A) = \frac{1}{4}\sum_{i=1}^{4}\left[1 - \left(\frac{1}{n_{x_i}}\sum_{j=1}^{n_{x_i}}(|A_j^{x_i^U} - 0.5| + |A_j^{x_i^L} - 0.5|)\right)^2\right].$$

- *For* E_L*, the function* $g(a) = a^2$ *is selected, which satisfies the properties of Corollary 40. The local lack of knowledge entropy obtained is given as:*

$$E_L(A) = \frac{1}{4}\sum_{i=1}^{4}\left(\sum_{j=1}^{n_{x_i}} S(A_j^{x_i})\right)^2.$$

- *For* E_H*, the function* $g(a) = 1 - \dfrac{1}{a^2}$ *is selected, which satisfies the properties of Corollary 47. The local hesitance entropy obtained is given as:*

$$E_H(A) = \frac{1}{4}\sum_{i=1}^{4}\left(1 - \frac{1}{n_{x_i}^2}\right).$$

Furthermore, let us select as a decision making criteria $D_H(A_i, \overline{\{1\}})$ *for each alternative, where* D_H *is the Hamming dissimilarity given in Example 26 and* $\overline{\{1\}}$ *represents the ideal alternative:*

$$D_H(A, \overline{\{1\}}) = \frac{1}{4}\sum_{i=1}^{4} \frac{1}{2n_{x_i}} \sum_{j=1}^{n_{x_i}} (2 - A_j^{x_i^U} - A_j^{x_i^L}).$$

Once that the three mappings and the criteria are defined, the value of each one has been obtained for the four interval-valued hesitant fuzzy sets, as it is shown in the Table 1.

Table 1
Different entropy values for four interval-valued hesitant fuzzy sets.

THFS(X)	A_1	A_2	A_3	A_4
x_1	{1}	{[0,0.4],[0.41,0.8],[0.81,1]}	{0.5}	{0}
x_2	{0}	{[0,0.4],[0.41,0.7],[0.71,1]}	{[0.45,0.5]}	{[0,0.004], [0.005,1]}
x_3	{0}	{[0,0.5],[0.51,0.7],[0.71,1]}	{[0.5,0.55], [0.56, 0.6]}	{[0.99,0.994], [0.995,1]}
x_4	{1}	{[0,0.5],[0.51,1]}	{[0.4,0.6]}	{1}
E_F	0	0.9032	0.9995	0.0197
E_L	0	0.9653	0.0126	0
E_H	0	0.8542	0.1875	0.375
$D_H(A_i, \overline{\{1\}})$	0.5	0.4552	0.4931	0.4383

The decision making criteria leads to the preference $A_4 > A_2 > A_3 > A_1$. *However, their values are really close, so another features can be taken into account. Depending on the aim of the business, each mapping of the entropy represents certain characteristic of interest. Let us analyze each situation separately:*

- A_1*: which is a crisp set, as the only values that it takes are* 0 *and* 1. *As a result, all the entropies are null, i.e.,* $(E_F, E_L, E_H) = (0, 0, 0)$, *because the only values are* 0 *and* 1 *(*$E_F(A_1) = 0$*), they are singletons (*$E_L(A_1) = 0$*) and there is a single interval (point) for each* x_i *(*$E_H(A_1) = 0$*). This shows that it is possible to obtain a low value in all of them with the same set.*

- A_2*: whose values are all close to or include the point* 0.5 *(high value of* $E_F(A_2)$*), the membership functions are close to the interval* $[0, 1]$ *(high value of* $E_L(A_2)$*) and for each point there are several intervals defining the membership function (high value of* $E_H(A_2)$*). Hence, the values of all the entropies are high, showing that this is possible in the same set.*

- A_3*: the memberships include and are all close to the point* 0.5 *(high value of* $E_F(A_3)$*), the total lengths of the memberships are small (low value of*

$E_L(A_3)$), and the number of intervals are one in three out of the four elements (low value of $E_H(A_3)$).

- *A_4: the memberships are all close to the extremes 0 and 1 (low value of $E_F(A_4)$), the total lengths of the memberships are very small (low value of $E_L(A_4)$), and the number of intervals are one in two out of the four elements and two in two out of the four elements (low-midium value of $E_H(A_4)$).*

For instance, if it is important that the experts have a similar opinion, the most influential mapping of the joint entropy is E_H, as the lower the value, the smaller the number of different opinions with respect to each parameter x_i. Then the alternatives A_1 and A_3 would be preferred over A_2 or A_4.

It is also remarkable that the last two sets show that the three mappings do not usually take similar values as it happened in the sets A_1 (low values) and A_2 (high values). In A_3, $E_F(A_3)$ is much higher than the other two, whlie in A_4, it is $E_H(A_4)$ which takes a greater value.

The utility of the entropy measure depends on the aim of each particular situation. Given several alternatives with a similar value with respect to the selected criteria (such as the distance between the alternative and the ideal alternative), a specific feature can be more important than others when choosing the best alternative.

However, this study of different situations can not be done with other entropy definitions, as they do not allow to analyze as many characteristics as the joint definition given in this paper. Obviously, the uncertainty associated to the definition of entropy given by Farhadinia in [14], is included in our new proposal, as the first mapping of our definition (E_F) represents a similar concept.

In summary, this new approach allows to obtain the classical concept of entropy for other types of sets, which is the distance to a crisp set, as well as another two uncertainties, related to the distance to a fuzzy set and to an interval-valued fuzzy set, being up to the researcher the importance given to each one in the studied situation.

4 Conclusion

In this paper, a new definition of entropy measure in a interval-valued hesitant fuzzy environment has been designed. This definition is given by three mappings, where each one represents a different type of entropy, which is a reliable alternative to the usual definitions of entropy defined just by one mapping.

The first mapping represents the distance to a union of crisp sets (fuzziness, E_F), which is close to the classical interpretation of entropy in other kind of sets. The second and third ones stand for the distance to a union of fuzzy sets (lack of knowledge, E_L) and the distance to only one interval-valued fuzzy set (hesitance, E_H), concepts which has not been applied in the past to this type of sets, the interval-valued hesitant fuzzy sets.

From this definition, different results have been developed in order to obtain that mappings in a more simplified way than the original definitions, with functions which satisfy more manageable properties, while being able to obtain several options of entropy varying a single parameter. A final example has been carried out to show that these three mappings complement each other, detecting different situations of uncertainty by the combination of them.

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