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A hybrid construction method based on weight functions to obtain interval-valued fuzzy relations

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Interval-valued fuzzy sets are an extension of fuzzy sets and are helpful when there is not enough information to define a membership function. This paper studies the behavior of a construction method for an interval-valued fuzzy relation built from a fuzzy relation. The behavior of this construction method is analyzed depending on the used t-norms and t-conorms, showing that different combinations of them produce a big variation in the results. Furthermore, a hybrid construction method that considers weight functions and a smoothing procedure is also introduced. Among the different applications of this method, the detection of edges in images is one of the most challenging. Thus, the performance of the proposal in detecting image edges is tested, showing that the hybrid approach that combines weights and a smoothing procedure provides better results than the non-weighted methods. Copyright © 2015 John Wiley & Sons, Ltd.

Keywords: interval-valued fuzzy sets; fuzzy sets; edge images; edge detection

1. Introduction

Fuzzy logic was introduced by Zadeh in [1] as a solution to the need of representing certain properties that Boolean logic cannot deal with. Fuzzy logic is a system of reasoning to work with information, which could be partially possible. This characteristic is modeled with membership functions. Even though from a mathematical point of view, there is no problem with working with membership functions; in practice, their definition often depends on the knowledge provided by an expert. According to Bustince [2], when the domain expert does not have enough information to define the membership function, it is more appropriate to define the membership degree of each element of a fuzzy set by an interval. This extension of fuzzy sets, known as interval-valued fuzzy sets, was introduced by Sambuc in [3]. In this generalization, the membership degree is given by a closed subinterval of [0, 1].

Interval-valued fuzzy sets have been applied to many different domains such as medicine [4], decision-making [5] or image processing [6]. More concretely, this kind of construction methods is often applied to the detection of edges in grey scale images, which has its most important application in the medical field [7] and other branches of science [8]. The importance of image processing in several areas is proven by the huge amount of studies devoted to this topic, where different problems with different tools are considered (see, for instance, [9–12]).

The aim of this paper is twofold. First, it studied how the selection of different t-norms and t-conorms affects the construction of interval-valued fuzzy relations from fuzzy relations. The study shows how the relation between t-norms and t-conorms can be linked to the relation between the interval-valued fuzzy relations obtained from them. Second, a new construction method for interval-valued fuzzy relations is proposed. This method is based on adding weights to make the points closer to the one studied, which has a greater strength in the construction method than the ones that are not.

The remainder of the paper is structured as follows. The next section briefly presents some preliminary concepts. Section 3 details the study related to the use of different t-norms and t-conorms. In Section 4, the weighted construction method is developed. Finally, in Section 5, some experiments are presented, and in Section 6, the main conclusions of this work are highlighted.

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2. Preliminary concepts

This section is divided into three subsections. In the former, basic definitions about fuzzy sets are provided. In the second one, the usual construction method of interval-valued fuzzy relations from fuzzy relations is explained. The last subsection is focused on the application to image edge detection.

2.1. Fuzzy sets

First of all, it is necessary to explain the basic concepts about fuzzy sets in order to understand the construction methods and results described in the next sections. All these definitions can be found in a wide range of sources. A good reference, with detailed information, is [13]. The first needed definition is the one of fuzzy set.

Definition 2.1
Let X be a set. A fuzzy set A in X is given by the membership function

$$A : X \to [0, 1].$$

A fuzzy set can be denoted by $A = \{(x, A(x)) | x \in X\}$. Furthermore, $FS(X)$ denotes the set of all fuzzy sets in X.

Along this paper, fuzzy relations, which are a special case of fuzzy sets, deal with the information required in the construction. Fuzzy relations are defined in the succeeding texts.

Definition 2.2
Let X and Y be two sets. A fuzzy relation R is a fuzzy set on $X \times Y$.

Thus, a fuzzy relation can be denoted by $R = \{((x, y), R(x, y)) | x \in X, y \in Y\}$, where $R(x, y) \in [0, 1]$ represents the degree of strength of the relation between x and y. Furthermore, $FR(X, Y)$ represents the set of all fuzzy relations in $X \times Y$. Moreover, we can consider the standard order in $FR(X \times Y)$: for any $R_1, R_2 \in FR(X, Y)$, we say that $R_1 \leq R_2$ iff $R_1(x, y) \leq R_2(x, y), \forall (x, y) \in X \times Y$.

When X and Y are finite universes, any fuzzy relation R in $X \times Y$ can be expressed in a matricial way.

All our developments are built around the concepts of triangular norms and conorms (t-norms and t-conorms, respectively). Both definitions are presented in the next two results.

Definition 2.3
A function $T : [0, 1]^2 \to [0, 1]$ is a t-norm if $\forall x, y, z \in [0, 1]$ satisfies

T1 Commutativity: $T(x, y) = T(y, x)$,
T2 Associativity: $T(x, T(y, z)) = T(T(x, y), z)$,
T3 Monotonicity: $T(x, y) \leq T(x, z)$, whenever $y \leq z$,
T4 Neutral element: $T(x, 1) = x$.

Definition 2.4
A function $S : [0, 1]^2 \to [0, 1]$ is a t-conorm if $\forall x, y, z \in [0, 1]$ satisfies

S1 Commutativity: $S(x, y) = S(y, x)$,
S2 Associativity: $S(x, S(y, z)) = S(S(x, y), z)$,
S3 Monotonicity: $S(x, y) \leq S(x, z)$, whenever $y \leq z$,
S4 Neutral element: $S(x, 0) = x$.

Both t-norms and t-conorms are an important part of the fuzzy theory, and it is usual to take pairs of t-norms and t-conorms, which satisfy the next properties.

Definition 2.5
Given a t-norm T and a t-conorm S,

- S is the dual t-conorm of T when

$$S(x, y) = 1 - T(1 - x, 1 - y),$$

- T is the dual t-norm of S when

$$T(x, y) = 1 - S(1 - x, 1 - y),$$

- T and S are duals if one of the two expressions in the aforementioned texts holds, since both of them are equivalent.

The next two definitions provide us a way to relate two t-norms (or t-conorms) with an order relation.

Definition 2.6
Let T_1, T_2 be two t-norms (respectively, t-conorms). T_1 is said to be lower than or equal to T_2, and it is denoted by $T_1 \leq T_2$, if and only if $\forall x, y \in [0,1]$, $T_1(x,y) \leq T_2(x,y)$.

Definition 2.7
Let T_1, T_2 be two t-norms (respectively, t-conorms). T_1 is said to be strictly lower than T_2, and it is denoted by $T_1 < T_2$, if and only if $T_1 \leq T_2$ and $\exists x_0, y_0 \in [0,1]$, such that $T_1(x_0,y_0) < T_2(x_0,y_0)$.

There are some well-known t-norms and t-conorms, which can be found in several sources, like in [14]. The most usual t-norms and t-conorms (dual pairs) are

- $T_M(x,y) = \min(x,y)$,
- $T_P(x,y) = xy$,
- $T_L(x,y) = \max(x+y-1, 0)$,
- $S_M(x,y) = \max(x,y)$,
- $S_P(x,y) = x+y-xy$,
- $S_L(x,y) = \min(x+y, 1)$,

where T_M, T_P, T_L (S_M, S_P, S_L) denote the t-norms (t-conorms) minimum (maximum), product and Lukasiewicz, respectively. It is known in [14] that $T_L < T_P < T_M$ and $S_M < S_P < S_L$.

There also exist t-norms and t-conorms families, from which most of the usual t-norms and t-conorms can be obtained depending on the chosen parameters. The most known ones are Frank t-norms [15], Yager t-norms [16], Dombi t-norms [17] and Sugeno-Weber t-norms [18, 19].

To complete this subsection, the definition of interval-valued fuzzy sets is given, as it is the main tool used in the paper in order to build the construction methods. This definition is an extension of the classical fuzzy sets given in Definition 2.1.

Definition 2.8
Given the set of all closed subintervals of [0, 1],

$$L([0,1]) = \left\{[a_l, a_u] | (a_l, a_u) \in [0,1]^2 \text{ and } a_l \leq a_u\right\},$$

then A is an interval-valued fuzzy set in the set X when it is defined by the membership function

$$A : X \to L([0,1]),$$

which can be decomposed as $A = [A_l, A_u]$, where A_l, A_u are fuzzy sets. An interval-valued fuzzy set can be denoted as $A = \{(x, A_l(x), A_u(x)) | x \in X\}$.

In the same way as it has been done with the classical fuzzy sets, the definition of interval-valued fuzzy sets is extended to interval-valued fuzzy relations.

Definition 2.9
Let X and Y be two sets. An interval-valued fuzzy relation R is an interval-valued fuzzy set in $X \times Y$.

An interval-valued fuzzy relation can be denoted by $R = \{((x,y), R(x,y)) | x \in X, y \in Y\}$, where $R(x,y)$ is the interval that measures the degree of strength of the relation between x and y.

2.2. *Construction method of interval-valued fuzzy relations*

This construction method builds an interval-valued fuzzy relation, where the starting point is a fuzzy relation, as it has been previously stated. This process is carried out with two constructors (lower and upper constructors) in order to obtain both sides of each interval with the values of each new fuzzy relation. From this interval-valued fuzzy relation, another fuzzy relation is defined as the length of each interval. This is the relation used to apply the method to generate fuzzy edge images.

Definition 2.10
Given two finite universes of natural numbers $X = \{0, 1, \ldots, P-1\}$ and $Y = \{0, 1, \ldots, Q-1\}$, $R \in FR(X,Y)$, a fuzzy relation in $X \times Y$, two t-norms T_1, T_2, two t-conorms S_1, S_2 and $n, m \in \mathbb{N}$, such that $n \leq \frac{P-1}{2}$ and $m \leq \frac{Q-1}{2}$,

- the lower constructor associated to T_1, T_2, n and m is given as follows:

$$L^{n,m}_{T_1,T_2} : FR(X,Y) \to FR(X,Y), \quad \text{where}$$
$$L^{n,m}_{T_1,T_2}[R](x,y) = \mathop{\overset{m}{\overset{n}{T_1}}}_{\substack{i=-n \\ j=-m}} \left(T_2(R(x-i, y-j), R(x,y))\right),$$

- the upper constructor associated to S_1, S_2, n and m is given as follows:

$$U^{n,m}_{S_1,S_2} : FR(X,Y) \to FR(X,Y), \quad \text{where}$$
$$U^{n,m}_{S_1,S_2}[R](x,y) = \mathop{\overset{m}{\underset{j=-m}{\overset{n}{\underset{i=-n}{S_1}}}}} (S_2(R(x-i,y-j),R(x,y))),$$

$\forall (x,y) \in X \times Y$, where i,j take values such that $0 \le x-i \le P-1$ and $0 \le y-j \le Q-1$, n and m indicate that the considered window is a matrix of dimension $(2n+1) \times (2m+1)$ and $\overset{n}{\underset{i=1}{T}} x_i = T(x_1,\dots,x_n)$.

The specific subsets of the natural numbers X and Y are considered in the previous definition, because the main application taken into account in this paper, of this method, is the edge image detection.

Definition 2.11
Let R be a fuzzy relation in $X \times Y$, $L^{n,m}_{T_1,T_2}[R]$ a lower constructor and $U^{n,m}_{S_1,S_2}[R]$ an upper constructor, then $R^{n,m}$ defined by

$$R^{n,m}_{T_1,T_2,S_1,S_2}(x,y) = \left[L^{n,m}_{T_1,T_2}[R](x,y), U^{n,m}_{S_1,S_2}[R](x,y)\right],$$

for all $(x,y) \in X \times Y$ is an interval-valued fuzzy relation in $X \times Y$.

In the previous definition, when S_i is the dual t-conorm of T_i, the interval-valued fuzzy relation is just denoted by $R^{n,m}_{T_1,T_2}$.
After obtaining both lower and upper constructors from the initial fuzzy relation (Definition 2.10), and the interval-valued fuzzy relation generated by them (Definition 2.11), the last step of the construction method is to obtain another fuzzy relation from such interval-valued fuzzy relation. To do so, the next definition is given, where the length of each interval is used.

Definition 2.12
Let R be a fuzzy relation in $X \times Y$ and let $L^{n,m}_{T_1,T_2}[R]$ and $U^{n,m}_{S_1,S_2}[R]$ be its lower and upper constructors, respectively; for two t-norms T_1 and T_2 and two t-conorms S_1 and S_2, the W-fuzzy relation associated to them is given by the following:

$$W\left[R^{n,m}_{T_1,T_2,S_1,S_2}\right](x,y) = U^{n,m}_{S_1,S_2}[R](x,y) - L^{n,m}_{T_1,T_2}[R](x,y).$$

Definitions 2.10, 2.11 and 2.12 [6] establish the construction method procedure, as it is schematized in Algorithm 1.

Algorithm 1 Non-weighted construction method algorithm.

Input: $R \in FR(X,Y)$, $n,m \in \mathbb{N}$, T_1, T_2 t-norms, S_1, S_2 t-conorms
Output: W-fuzzy relation $W \in FR(X,Y)$

1: Obtain the lower constructor $L^{n,m}_{T_1,T_2}[R]$ associated to n, m, T_1 and T_2 (Def. 2.10)
2: Obtain the upper constructor $U^{n,m}_{S_1,S_2}[R]$ associated to n, m, S_1 and S_2 (Def. 2.10)
3: Construct the interval-valued fuzzy relation $R^{n,m}_{T_1,T_2,S_1,S_2}$ from $L^{n,m}_{T_1,T_2}$ and $U^{n,m}_{S_1,S_2}$ (Def. 2.11)
4: Obtain the W-fuzzy relation $W[R^{n,m}_{T_1,T_2,S_1,S_2}]$ from $R^{n,m}_{T_1,T_2,S_1,S_2}$ (Def. 2.12)

2.3. The problem of edges detection in grey scale images

The application that will be studied in this paper is the detection of edges in grey scale images. The detection of edges in images has one of its most important application in the medical field, where it can be used, for example, for brain tumor pattern recognition [7].
In order to adapt the previous construction method, it is necessary to explain how to deal with grey scale images and their representation.

Definition 2.13
A grey scale image R whose dimensions are $P \times Q$ pixels is a fuzzy relation where the finite sets used are $X = \{0,1,\dots,P-1\}$ and $Y = \{0,1,\dots,Q-1\}$.

This means that the grey scale images are represented by fuzzy relations. With this premise, all the construction methods can be applied, and the outputs are the following:

- *The lower constructor.* It represents a darker version of the original image. Depending on the t-norms chosen, this image can be more or less dark. In Figure 1, there is a representation of three lower constructors with different pairs of t-norms.

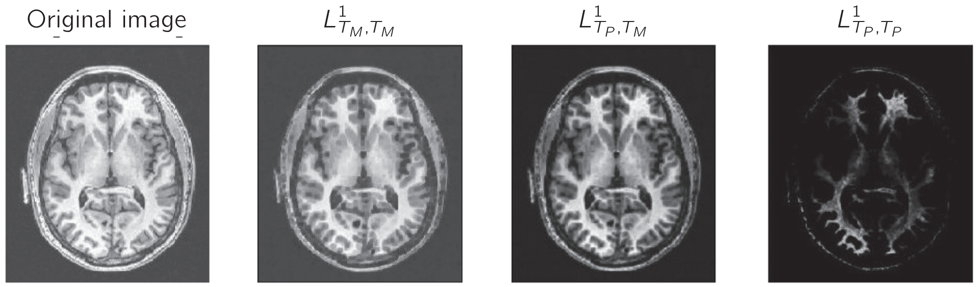


Figure 1. Comparison of lower constructors depending on the used t-norms, where T_M and T_P are the minimum and product t-norms, respectively.

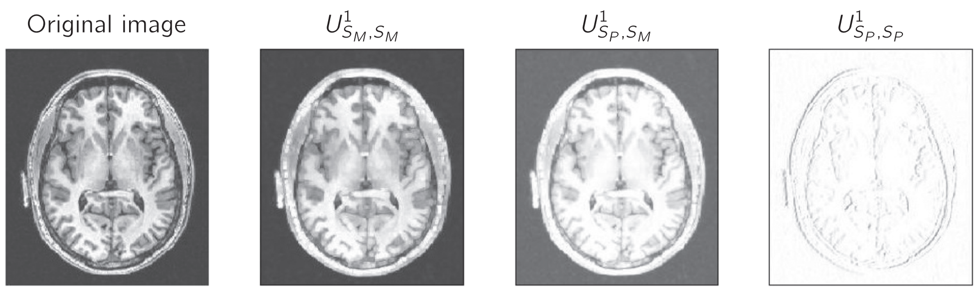


Figure 2. Comparison of upper constructors depending on the used t-conorms, where S_M and S_P are the maximum and product t-conorms, respectively.

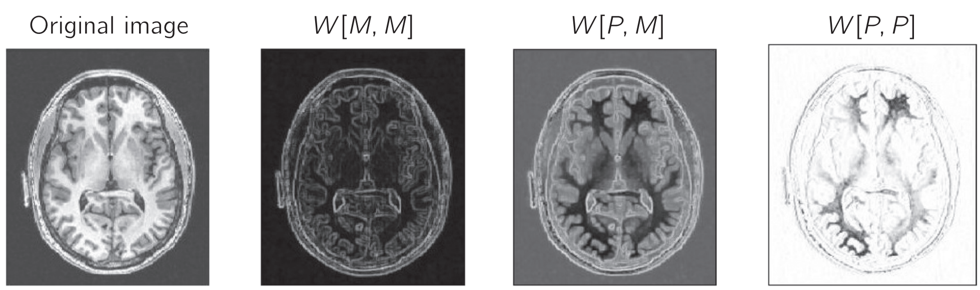


Figure 3. Comparison of W-fuzzy images depending on the used pairs of t-norms and t-conorms, where $W[P, M] = U^{n,m}_{S_P,S_M} - L^{n,m}_{T_P,T_M}$, and analogously for the others.

- *The upper constructor.* It represents a brighter version of the original image. Depending on the t-conorms chosen, this image can be more or less bright. In Figure 2, there is a representation of three upper constructors with different pairs of t-conorms.
- *The W-fuzzy edge image.* It represents the difference of contrast between both constructors. The edges can be identified in this image. In Figure 3, there is a representation of three W-fuzzy images with different pairs of t-norms and t-conorms.

Figures 1–3 highlight the fact that different t-norms and t-conorms cause a variation in the resulting lower constructor, upper constructor and W-fuzzy image, respectively. That is the reason to study, in the next section, certain properties that these relations keep from the selected t-norms and t-conorms.

3. Influence of the chosen t-norms and t-conorms

As it has been said in the previous section, the construction method needs two t-norms and two t-conorms, and as Figures 1–3 show, the selection affects the resulting interval-valued fuzzy relation. It seems a natural step to study how the relation between t-norms and t-conorms can be reflected in the constructors, and therefore, in the interval-valued fuzzy relation.

This section studies how the relations between two t-norms (resp. t-conorms) are translated to the constructors. In addition, some examples with the most usual t-norms (resp. t-conorms) are shown.

Proposition 3.1
Given four t-norms T_a, T_b, T_c, T_d, if $T_a \leq T_b$ and $T_c \leq T_d$, then

$$L^{n,m}_{T_a,T_c} \leq L^{n,m}_{T_b,T_d}.$$

Proof
Let R be any fuzzy relation in $X \times Y$. Taking into account the t-norms monotony property,

$$\begin{aligned} L^{n,m}_{T_a,T_c}[R](x,y) &= \mathop{T_a}_{\substack{i=-n\\ j=-m}}^{\substack{m\\ n}} (T_c(R(x-i,y-j),R(x,y))) \leq \\ &\leq \mathop{T_a}_{\substack{i=-n\\ j=-m}}^{\substack{m\\ n}} (T_d(R(x-i,y-j),R(x,y))) \leq \\ &\leq \mathop{T_b}_{\substack{i=-n\\ j=-m}}^{\substack{m\\ n}} (T_d(R(x-i,y-j),R(x,y))) = L^{n,m}_{T_b,T_d}[R](x,y). \end{aligned}$$

□

The same relation is satisfied for t-conorms as the next proposition states.

Proposition 3.2
Given four t-conorms S_a, S_b, S_c, S_d, if $S_a \leq S_b$ and $S_c \leq S_d$, then

$$U^{n,m}_{S_a,S_c} \leq U^{n,m}_{S_b,S_d}.$$

After these results about both lower and upper constructors, the next step is to analyze what happens with W-fuzzy relations.

Corollary 3.3
Given four pairs of dual t-norms and t-conorms (T_a, S_a), (T_b, S_b), (T_c, S_c) and (T_d, S_d), where $T_a \leq T_b$ and $T_c \leq T_d$, and because of the duality, $S_a \geq S_b$ and $S_c \geq S_d$, then

$$W[R^{n,m}_{a,c}] \geq W[R^{n,m}_{b,d}].$$

Proof
It is immediate, since

$$W\left[R^{n,m}_{a,c}\right](x,y) = U^{n,m}_{S_a,S_c}[R](x,y) - L^{n,m}_{T_a,T_c}[R](x,y) \geq U^{n,m}_{S_b,S_d}[R](x,y) - L^{n,m}_{T_b,T_d}[R](x,y) = W\left[R^{n,m}_{b,d}\right](x,y)$$

for any $(x,y) \in X \times Y$. □

The last result is proven straightforwardly from the previous results about t-norms and t-conorms. Keeping this last corollary in mind, it is possible to apply this to some particular cases with well-known t-norms and t-conorms, as it is shown in the next examples.

Example 3.4 (Minimum-maximum, product and Lukasiewicz)
Because $T_L < T_P < T_M$ and $S_M < S_P < S_L$, the W-fuzzy relations are related as given in Figure 4, where $W[M,P]$ denotes $W\left[R^{n,m}_{T_M,T_P}\right] = U^{n,m}_{S_M,S_P}[R] - L^{n,m}_{T_M,T_P}[R]$ for any $R \in FR(X,y)$, and analogously for the others.

Furthermore, in order to see the results in the edge image detection, in Figure 5, these nine combinations of W-fuzzy relations are shown for a grey scale image. It is easy to see that the results get reflected in this application. From this figure, it can be noted that the use of certain t-norms and t-conorms builds an unclear W-fuzzy relation, like Lukasiewicz ones. This is the reason to skip them in the carried out experimentation.

Example 3.5 (Frank t-norms [15])
Taking into account the definition of this family, with $\lambda \in [0,\infty]$,

$$T^F_\lambda(x,y) = \begin{cases} T_M(x,y), & \text{if } \lambda = 0, \\ T_P(x,y), & \text{if } \lambda = 1, \\ T_L(x,y), & \text{if } \lambda = \infty, \\ log_\lambda\left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right), & \text{in other case,} \end{cases}$$

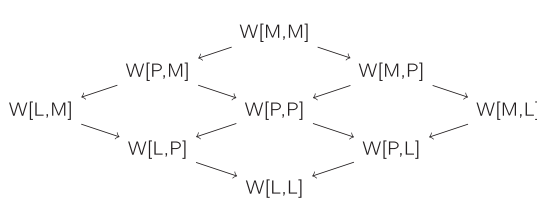


Figure 4. Relationships between the different W-fuzzy relations depending on t-norms and t-conorms.

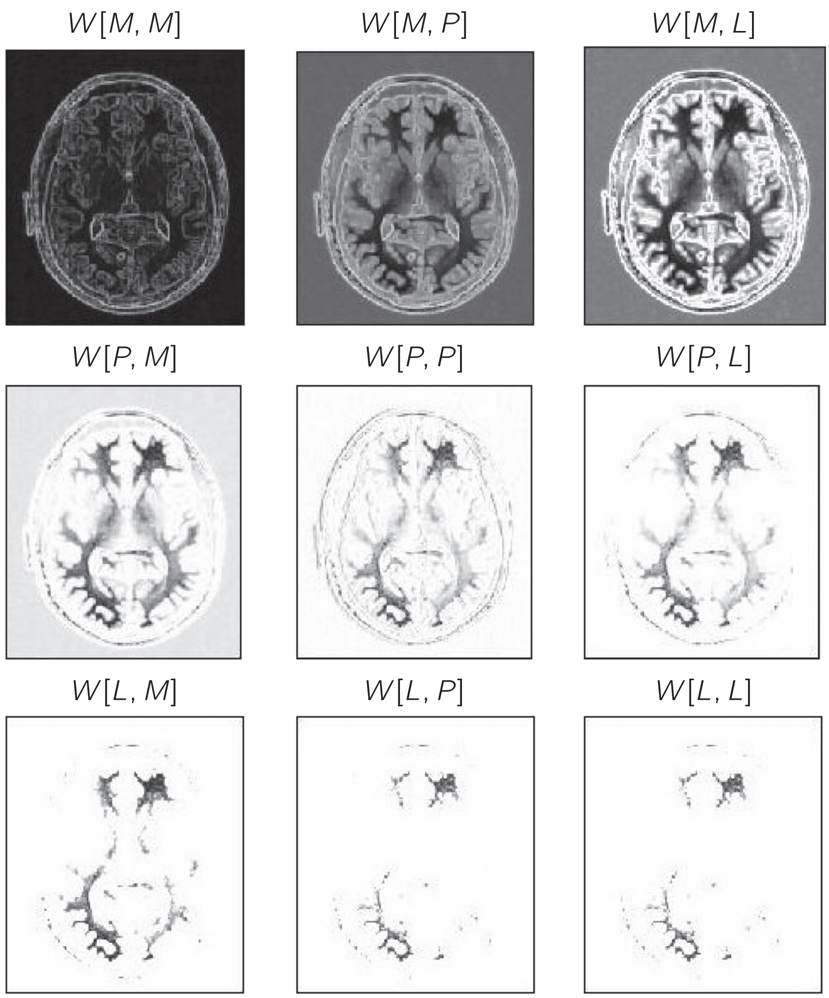


Figure 5. W-fuzzy edge images obtained by the combination of minimum-maximum, product and Lukasiewicz t-norms and t-conorms.

$$S^F_\lambda(x,y) = \begin{cases} S_M(x,y), & \text{if } \lambda = 0, \\ S_P(x,y), & \text{if } \lambda = 1, \\ S_L(x,y), & \text{if } \lambda = \infty, \\ 1 - log_\lambda\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right), & \text{in other case,} \end{cases}$$

and given $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $\lambda_1 > \lambda_2$ and $\lambda_3 > \lambda_4$, then

$$\begin{array}{ll} T^F_{\lambda_1} \leq T^F_{\lambda_2} & S^F_{\lambda_1} \geq S^F_{\lambda_2}, \\ T^F_{\lambda_3} \leq T^F_{\lambda_4} & S^F_{\lambda_3} \geq S^F_{\lambda_4}, \end{array}$$

and therefore, $L^{n,m}_{T^F_{\lambda_1},T^F_{\lambda_3}} \leq L^{n,m}_{T^F_{\lambda_2},T^F_{\lambda_4}}$ and $U^{n,m}_{S^F_{\lambda_1},S^F_{\lambda_3}} \leq U^{n,m}_{S^F_{\lambda_2},S^F_{\lambda_4}}$. As a result, $W\left[\lambda^F_2, \lambda^F_4\right] \leq W\left[\lambda^F_1, \lambda^F_3\right]$.

These results can also be adapted to other families of t-norms and t-conorms like Yager, Dombi or Sugeno-Weber families.

All these results provided in this section are useful for choosing the right combination of t-norms and t-conorms depending on the purpose of the study, as the greater the value of the W-fuzzy image, the brighter the pixels in the image, and therefore, affecting the appearance of the edges in such image.

4. Weighted construction method

In the method developed by Barrenechea *et al.* [6] for each element of the relation, a centered window in that element is considered. Window dimension depends on natural numbers *n* and *m*, which are the parameters in both constructors.

This new approach tries to make that the closer the value to the center of the window, the greater the importance that it takes in the definition of the constructors. This goal is useful, as the detection of an edge must be more related to the pixels that are closer to the central one, and that is the main objective of this section: to develop a method to capture this reasoning.

In other words, our goal is to obtain the final lower and upper constructors, which are obtained by weighting the original lower and upper constructors given in Definition 2.10 by means of weights, in such a way that the smaller windows have more strength in the definition. Formally,

Definition 4.1
Given two finite universes of natural numbers $X = \{0, 1, \ldots, P-1\}$ and $Y = \{0, 1, \ldots, Q-1\}$, $R \in FR(X, Y)$ a fuzzy relation in $X \times Y$, two t-norms T_1, T_2, two t-conorms S_1, S_2 and $n, m \in \mathbb{N}$ such that $n \leq \frac{P-1}{2}$ and $m \leq \frac{Q-1}{2}$, for any $i = 1, 2 \ldots, \max(n, m)$, we can consider the two fuzzy relations $L^i[R]$ and $U^i[R]$ defined by

$$L^i[R](x,y) = L_{T_1,T_2}^{\min(i,n),\min(i,m)}[R](x,y)$$

and

$$U^i[R](x,y) = U_{S_1,S_2}^{\min(i,n),\min(i,m)}[R](x,y).$$

The next step has to be to weight these values in an appropriate way. Thus, we obtain the final lower and upper constructors associated to any fuzzy relation $R \in FR(X \times Y)$ as follows:

$$L[R](x,y) = \sum_{i=1}^{k} w_i L^i[R](x,y) \quad \text{and} \quad U[R](x,y) = \sum_{i=1}^{k} w_i U^i[R](x,y),$$

where k denotes the maximum of n and m.

Finally, it is necessary to determine the weights w_i such that they satisfy $w_i \geq w_{i+1}$, so the smaller windows have more strength. The weights proposed in this paper are based on assigning a different value to each point in the window. We have considered the following three cases:

- Average of the k windows,

$$\begin{cases} \sum_{i=1}^{k} w_i = 1, \\ w_1 = \cdots = w_k \in (0,1), \; i = 1, \ldots, k, \end{cases}$$

  which leads us to the weights

$$w_i = \frac{1}{k}, \quad i = 1, \ldots, k. \tag{1}$$

- An equidistant version of the weights with a constant increase given in the next restrictions

$$\begin{cases} \sum_{i=1}^{k} w_i = 1, \\ w_i \in (0,1), \qquad i = 1, \ldots, k, \\ w_i - w_{i+1} = C, \; i = 1, \ldots, k-1, \\ w_k = C, \end{cases}$$

  where $C \in (0, 1)$ is a constant. With some calculation, the expression of the weights is reached as follows:

$$w_{k-1} - w_k = C \Rightarrow w_{k-1} = 2C \Rightarrow \cdots \Rightarrow w_{k-i} = (i+1)C,$$

  and applying it on the first condition

$$1 = \sum_{i=1}^{k} w_i = \sum_{i=0}^{k-1} w_{k-i} = \sum_{i=0}^{k-1} (i+1)C = C \sum_{i=0}^{k-1} (i+1) = C\frac{k(k+1)}{2}.$$

  Hence, $C = \frac{2}{k(k+1)}$ and the weights are

$$w_{k-i} = \frac{2(i+1)}{k(k+1)}, \quad i = 0, \ldots, k-1,$$

  or equivalently,

$$w_i = \frac{2(k-i+1)}{k(k+1)}, \quad i = 1, \ldots, k. \tag{2}$$

- A constant relation between two consecutive weights, given $N \in \mathbb{N}\setminus\{1\}$

$$\begin{cases} \sum_{i=1}^{k} w_i = 1, \\ w_i \in (0,1), \quad i = 1,\dots,k, \\ w_i/w_{i+1} = N,\ i = 1,\dots,k-1, \\ w_k = C, \end{cases}$$

where $C \in (0,1)$ is a constant. From these conditions, it follows that $w_i = N^{k-i}C$ for all i. Thus, after some calculations, the expression of the weights is reached as follows:

$$1 = \sum_{i=1}^{k} w_i = C\sum_{i=1}^{k} N^{k-i} = C\sum_{i=0}^{k-1} N^{i} = C\frac{N^k-1}{N-1} \Rightarrow C = \frac{N-1}{N^k-1} \Rightarrow \\ \Rightarrow w_i = N^{k-i}\frac{N-1}{N^k-1},\ i = 1,\dots,k. \tag{3}$$

Note that the bigger the value of N, the greater the importance on the central pixels. Moreover, note that the case $N = 1$ is not considered, because in that case, the weights are selected according to the average of the k windows method.

Once the values of the weights are calculated, and therefore, both lower and upper constructors, the remaining steps of the method in Section 2 (Definitions 2.11 and 2.12) must be applied in order to get the new interval-valued fuzzy relation and the W-fuzzy relation.

It should be noted that the fact of using weights causes the appearance of some values very close to 0 or 1, but not the own value. The reason is that the use of the biggest windows can make a little influence in such value. To avoid this situation, a smoothing step is used such that the final W-fuzzy edge image W is modified with some cut point $\alpha \in (0, 0.5)$.

Given W a fuzzy relation, the smoothing step with cut point $\alpha \in (0, 0.5)$ is carried out as follows:

1. If $W(x,y) < \alpha$, then its value is modified such that $W^{\alpha}(x,y) = 0$.
2. If $W(x,y) > 1-\alpha$, then its value is modified such that $W^{\alpha}(x,y) = 1$.
3. For the remaining values in the closed interval $[\alpha, 1-\alpha]$, they are expanded to the closed interval $[0,1]$, keeping the original proportion

$$W(x,y) \to W^{\alpha}(x,y) = 0.5 + \frac{1}{1-2\alpha}(W(x,y) - 0.5).$$

The reason to take values in the interval $(0, 0.5)$ lies in the fact that when $\alpha \to 0$, the smoothing step leads us to the method without such step, as sets of points modified by parts 1 and 2 of it tend to the empty set. Meanwhile, the remaining values get the modification

$$W^{\alpha}(x,y) = 0.5 + \frac{1}{1-2\alpha}(W(x,y) - 0.5) \xrightarrow[\alpha\to 0]{} 0.5 + \frac{1}{1}(W(x,y) - 0.5) = W(x,y).$$

If the value α is greater or equal to 0.5, it would make no sense to apply the smoothing step, as there would be pixels whose value must be changed to 0 (step 1) and to 1 (step 2) at the same time.

Algorithm 2 Weighted construction method algorithm.

Input: $R \in FR(X,Y)$, $n, m \in \mathbb{N}$, T_1, T_2 t-norms, S_1, S_2 t-conorms, $N \in \mathbb{N}$, $\alpha \in (0, 0.5)$
Output: W-fuzzy relation $W^{\alpha} \in FR(X,Y)$

1: Fix $k = max(n,m)$
2: Obtain $L^i[R]$ associated to n, m, T_1 and T_2 $\forall i = 1,\dots,k$ (Def. 4.1)
3: Obtain $U^i[R]$ associated to n, m, S_1 and S_2 $\forall i = 1,\dots,k$ (Def. 4.1)
4: Obtain weights w_i for $i = 1,\dots,k$, with the method assigned ((1), (2) or (3))
5: Calculate the lower and upper constructors as follows:

$$L[R](x,y) = \sum_{i=1}^{k} w_i L^i[R](x,y) \text{ and } U[R](x,y) = \sum_{i=1}^{k} w_i U^i[R](x,y)$$

6: Construct the interval-valued fuzzy relation $R^{n,m}_{T_1,T_2,S_1,S_2}$ from L and U (Def. 2.11)
7: Obtain the W-fuzzy relation $W\left[R^{n,m}_{T_1,T_2,S_1,S_2}\right]$ from $R^{n,m}_{T_1,T_2,S_1,S_2}$ (Def. 2.12)
8: Calculate W^{α} from W with the smoothing step defined by the cut point α

The scheme of the weighted method with the smoothing step is shown in Algorithm 2. In order to obtain the weighted method without the smoothing step, point 8 of Algorithm 2 is skipped, where $W = W_w$.

In the experimentation carried out in the next section, the influence of α, along with the comparison of this new approach with the non-weighted one, is analyzed with a grey scale database.

5. Experiments

In this section, the weighted method is compared with the one introduced in [6]. To do that, a grey scale images database has been considered. These grey scale images were obtained from the Berkeley Segmentation Dataset [20]. This database contains original images and its corresponding edge images, which are used as the base to the comparison between all the methods of study.

The first 25 images from the test set were selected, whose dimensions are 481×321 (or 321×481) pixels. The t-norms and t-conorms selected for this study are the standard ones (Minimum-Maximum), as the goal of this experimentation is to check if the weighted method outperforms the non-weighted one under the same conditions.

The studied situations in this experimentation are as follows:

- Non-weighted method with $n = m = 1$ (windows of size 3×3) (NW1).
- Non-weighted method with $n = m = 2$ (windows of size 5×5) (NW2).
- Weighted method with the combination of
  - *Methods to obtain weights:* average (A), equidistant method (I) and constant relation method with $N = 2, 3, 4, 5$ (II, III, IV, V).
  - *Number of terms:* $k = 2$, $k = 3$ or $k = 4$ terms (2,3,4).
  - *Smoothing step parameter value:* $\alpha \in \{0, 0.05, 0.1, \dots, 0.4, 0.45\}$.

Note that the equidistant method with 2 terms and the constant relation method with $N = 2$ with 2 terms too are the same. The non-weighted selected situations are the ones that obtain the best results (as it has been proven in [6]), and that is the reason to select these window sizes. In Figure 6, some of the W-fuzzy images obtained for each one of the test images are given.

Each edge image is compared with the one given by the source. To make such comparison, for each image, a value is assigned. Let S be the edge image given in [20] and let M be the one obtained by our method; the value assigned to M is given by the following expression:

$$v(M) = \frac{1}{\#(X \times Y)} \sum_{(x,y) \in X \times Y} |M(x, y) - S(x, y)|,$$

where $\#(X \times Y)$ represents the number of pixels of the image.

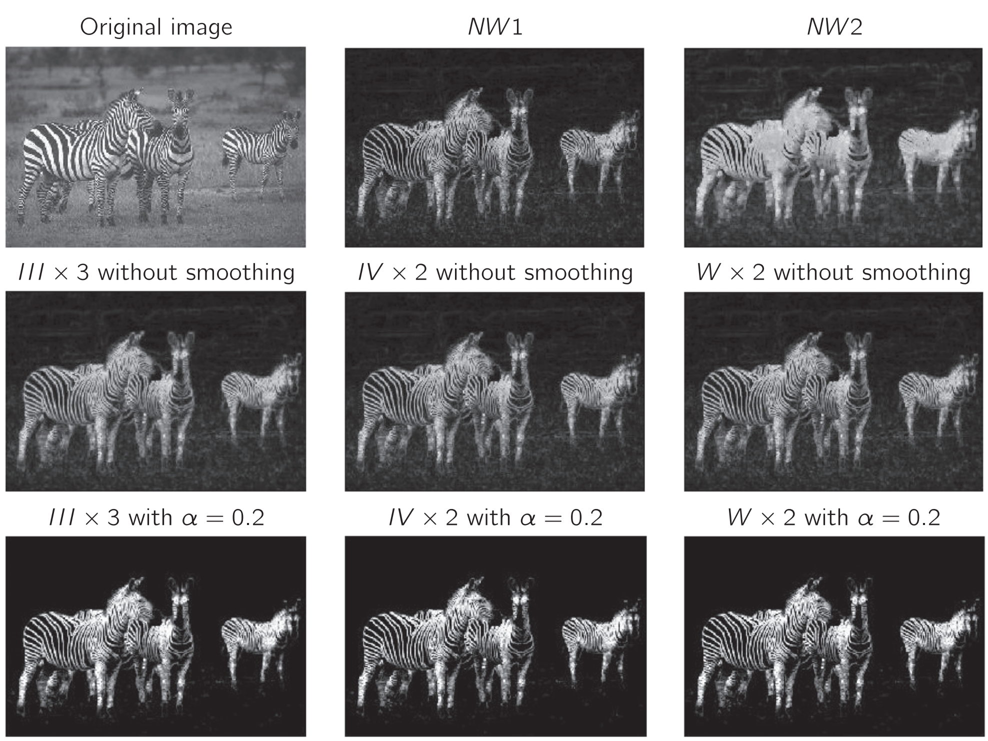


Figure 6. Comparison of W-fuzzy images depending on the experimental parameters, where two non-weighted methods ($NW1$, $NW2$), and $III \times 3$, $IV \times 2$ and $V \times 2$ without smoothing step and with $\alpha = 0.2$ are shown.

Notice that this value $v(M)$ is in fact the normalized Hamming distance between M and the edge image S (see, for instance, [13]), because fuzzy relations are just fuzzy sets of $X \times Y$.

The parameter α that defines the smoothing step is an important factor that must be taken into account when comparing the results. In Figure 7, five representations of an image with different values of α are presented.

To summarize the results obtained, the mean of the value $v(M)$ for all the selected 25 images is calculated for each method. Obviously, the smaller mean value provides the best result.

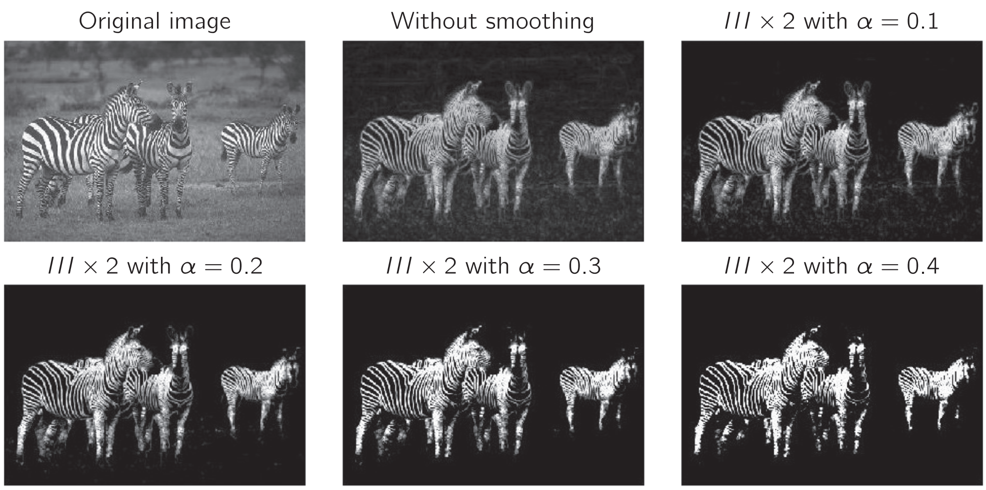


Figure 7. Comparison of W-fuzzy images depending on the parameter α, where it used the constant relation method to obtain the weights with two terms ($III \times 2$).

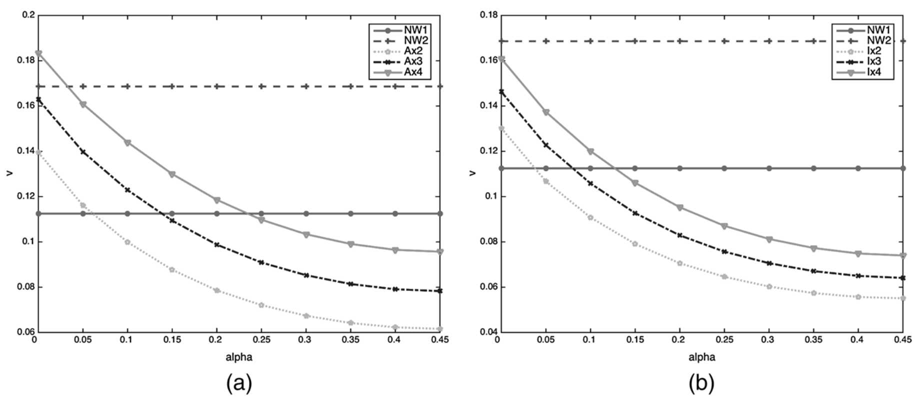


Figure 8. (a) Non-weighted methods (NW1, NW2) and weighted methods with average weights (A). (b) Non-weighted methods (NW1, NW2) and weighted methods with equidistant weights (I).

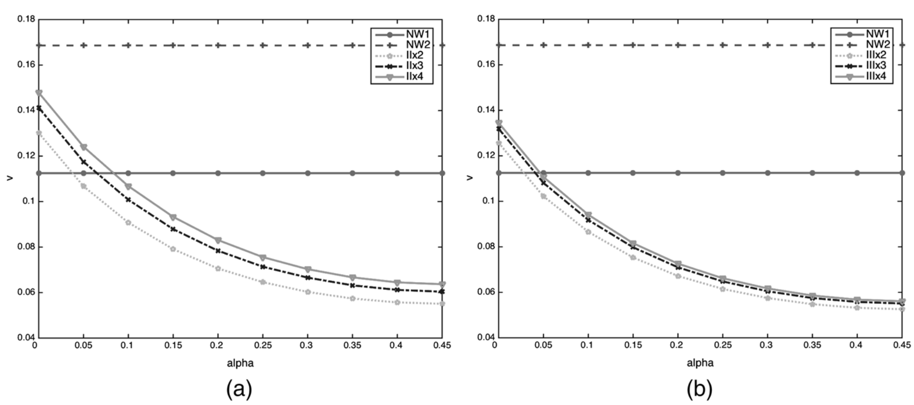
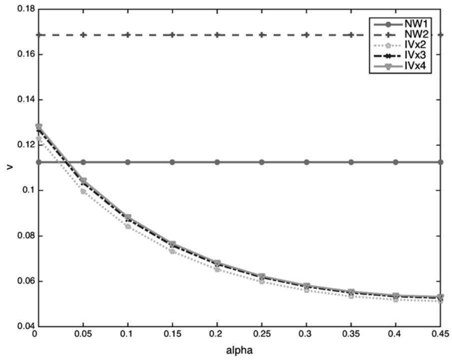


Figure 9. (a) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 2$ (II). (b) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 3$ (III).



(a) (b)

Figure 10. (a) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 4$ (IV). (b) Non-weighted methods (NW1, NW2) and weighted methods with constant relation weights for $N = 5$ (V).

Figures 8–10 provide the mean values of $v(M)$ obtained for each method, where X and Y axes represent the value α of the smoothing step and the mean value obtained for that method, respectively.

From the results shown in Figures 8–10, all the methods, but one in the first graphic, always obtain better results than the non-weighted method with $m = n = 2$ (NW2). Meanwhile, the other non-weighted method where $m = n = 1$ (NW1) is also improved by every combination of type of weights and number of terms, from an α value onwards. Depending on the method analyzed, this α value can be further or closer to 0, as it can be observed in the graphics.

These results prove that this new method, where weights are added to the construction method, overcomes the one without its use, as long as the smoothing step is applied with a big enough α value.

6. Conclusions

This paper presents a method to construct interval-valued fuzzy relations where the starting point is a fuzzy relation. The method is based on introducing weights to make that the closer the value to the point under study, the greater the importance that it takes in the definition of the constructors. The behavior of this construction method has been analyzed depending on the combination of t-norms and t-conorms used to define the interval, showing that different combinations of t-norms (resp. t-conorms) produce divergent outputs. In addition to that, it has been proven that some relations that hold between two t-norms (resp. t-conorms) are transferred to the constructors that shape the method, and therefore, the final edge detection also depends on it.

Finally, a smoothing process is also considered in order to erase small variations in the extremes produced by the biggest window with small weights. Through the experimentation, it is shown that the hybrid approach, which combines both weights and the smoothing step, provides better results than the non-weighted methods, where the parameter of the smoothing step depends on the type of weights and number of terms.

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